

Time Dependent Quantum Mechanics

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Abstract

We present a systematic method for dealing with time dependent quantum dynamics, based on the quantum brachistochrone and matrix mechanics. We derive the explicit time dependence of the Hamiltonian operator for a number of constrained finite systems from this formalism. Once this has been achieved we go on to calculate the wavevector as a function of time, in order to demonstrate the use of matrix methods with respect to several concrete examples. Interesting results are derived for elliptic curves and periodic orbits on higher dimensional non-commutative geometries.

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1 Introduction

Recent developments in quantum mechanical theory have focused on the time-dependent dynamical problem, which proposes the use of Floquet theory in order to derive the time evolution operators for the dependent system. In our paper, we consider a new set of time-dependent quantum mechanical operators which define the physics of finite systems. Standard quantum mechanics postulates the existence of a Hamiltonian operator \tilde{H} and an initial state $|\Psi(0)\rangle$ and seeks to extrapolate to find our final state $|\Psi(t)\rangle$, via the von Neumann equation. This is an initial value problem; we wish to find and calculate a boundary value problem that is consistent with quantum physics. This has already been calculated in the papers of Carlini et. al [1]. The principal methodology is to construct a time-optimal Hamiltonian matrix which moves us from state-to-state on the projective manifold, in least time. These paths are geodesic flows; they define the surface of the state space in the same way that the lines of longitude on the sphere define the globe itself. In addressing this question, we are naturally led to the analysis of explicitly time-dependent quantum systems. These dynamical systems are generally periodic in nature for the examples we have been able to calculate; this periodicity allows us to use Floquet theory in order to find

solutions for the problems in question. In considering these types of general dynamical problems we find a number of time dependent states with constant energies; wave-vector analysis allows us to consider the geometry of the state space for higher-dimensional realms where it is no longer possible to explicitly graph the paths the state takes over time. Experiments have been conducted into these areas with the particular aim of creating these exotic qutrit states, as in [2], [3] and as such it is important that we be able to predict best methods of state control within this regime. Relevant references to the geometry of qutrits and SU(3) operators may be found in [4]. Our method differs from standard quantum mechanics in that we do not assume the form of the Hamiltonian operator; we derive it using the quantum brachistochrone hypothesis, being that the physical path is the one of least time.

2 Quantum Fermat Principle

This section reproduces the arguments of [1] in deriving the quantum brachistochrone equation from the Quantum Fermat Principle, being the hypothesis that the state vector travels along a path of least time on the complex projective manifold. These results are not new but are included in order to outline the fundamentals of this methodology. Firstly, let us consider the energy variance on the state space. We may write this as:

$$(\Delta E)^2 = \langle \psi | (\hat{H} - \langle \hat{H} \rangle)^2 | \psi \rangle \quad (1)$$

We may define the metric of the projective state space:

$$ds = \Delta E \cdot dt \quad (2)$$

The quantum state may be expanded in an orthonormal basis over a set of probabilities:

$$|\psi\rangle = \sum_j \sqrt{p_j} e^{i\varphi_j} |j\rangle \quad (3)$$

which then define the state space metric:

$$\frac{1}{4} ds^2 = 1 - |\langle \psi + d\psi | \psi \rangle|^2 = \langle d\psi | (\hat{\mathbf{1}} - \hat{P}) | d\psi \rangle \quad (4)$$

where $\hat{P} = |\psi\rangle\langle\psi|$, the standard pure state projection operator. For the time-dependent quantum control problem, the action principle must be recast in such a way that we can apply it to find optimal transfers between states. Our solution, where it exists, will be a time dependent Hamiltonian that takes us time optimally from one state to another. We do not rule out the existence of systems where there are a number of extremal trajectories; however, our principal aim to search for global extrema where the time is a minimum regardless of our particular choice of initial and final states. Writing out the expression for the

time optimal quantum control action we have:

$$S = \int_{t_0}^{t_f} (\mathcal{L}_T + \mathcal{L}_S + \mathcal{L}_C) dt \quad (5)$$

$$\mathcal{L}_T = \frac{\sqrt{\langle \dot{\psi} | (\hat{\mathbf{1}} - \hat{P}) | \dot{\psi} \rangle}}{\Delta E} = \frac{1}{\Delta E} \frac{ds}{dt} \quad (6)$$

$$\begin{aligned} \mathcal{L}_S &= i(\langle \phi | \dot{\psi} \rangle - \langle \dot{\psi} | \phi \rangle) - (\langle \phi | \hat{H} | \psi \rangle + \langle \psi | \hat{H} | \phi \rangle) \\ \mathcal{L}_C &= \sum_j \lambda_j f_j(\hat{H}) \end{aligned} \quad (7)$$

These equations are gauge invariant; we therefore make a particular choice of the gauge, being the reference frame in which the Hamiltonian is traceless:

$$\hat{H} \mapsto \hat{H} - \frac{\hat{\mathbf{1}}_N}{N} \text{Tr}(\hat{H}) = \tilde{H} \quad (8)$$

$$\text{Tr}(\tilde{H}) = 0 \quad (9)$$

The first term is unity:

$$\mathcal{L}_T = \frac{\Delta E}{\Delta E} = 1 \quad (10)$$

and therefore contributes a term $\int dt$ to the action, justifying our claim of time optimality. This is the quantum analogue of Fermat's principle of least time. Writing the Euler-Lagrange equations that result from the variations of the action integral, we find the system of equations:

$$i |\dot{\psi}\rangle = \tilde{H} |\psi\rangle \quad (11)$$

$$-\frac{i}{2} \frac{d}{dt} \left(\frac{\tilde{H} - \langle \tilde{H} \rangle \hat{\mathbf{1}}}{\Delta E^2} \right) |\psi\rangle = i |\dot{\phi}\rangle - \tilde{H} |\phi\rangle \quad (12)$$

$$\frac{(\{\tilde{H}, \hat{P}\} - 2 \langle \tilde{H} \rangle \hat{P})}{2\Delta E^2} + |\phi\rangle \langle \psi| + |\psi\rangle \langle \phi| = \sum_j \lambda_j \frac{\partial f_j}{\partial \tilde{H}} \quad (13)$$

$$f_j(\tilde{H}) = 0 \forall j \quad (14)$$

Defining our constraint operator:

$$\hat{A} = \sum_j \lambda_j \frac{\partial f_j}{\partial \tilde{H}} = \frac{(\{\tilde{H}, \hat{P}\} - 2 \langle \tilde{H} \rangle \hat{P})}{2\Delta E^2} + |\phi\rangle \langle \psi| + |\psi\rangle \langle \phi| \quad (15)$$

Taking traces and expectation values of the numerator of the first term in the constraint operator:

$$\text{Tr}(\{\tilde{H}, \hat{P}\} - 2 \langle \tilde{H} \rangle \hat{P}) = 0 \quad (16)$$

$$\langle (\{\tilde{H}, \hat{P}\} - 2\langle \tilde{H} \rangle \hat{P}) \rangle = 0 \quad (17)$$

We derive the fundamental result:

$$\langle \hat{A} \rangle = Tr(\hat{A}) = \langle \psi | \phi \rangle + \langle \phi | \psi \rangle \quad (18)$$

Computing the anticommutator of the A-operator and the projector we obtain:

$$\{\hat{A}, \hat{P}\} = \hat{A} + \langle \hat{A} \rangle \hat{P} \quad (19)$$

We now define an associated operator to specify the boundary condition equations:

$$\hat{G} = \hat{A} - \langle \hat{A} \rangle \hat{P} \quad (20)$$

$$\{\hat{G}(t), \hat{P}(t)\} = \hat{G}(t) \quad (21)$$

$$Tr(\hat{G}) = 2 \langle \hat{G} \rangle = 0 \quad (22)$$

We now perform some simple algebraic manipulations amongst our original Euler-Lagrange equations. Left-multiplying (11) with $\langle \phi |$ and the conjugate of (12) with $|\psi \rangle$ we find:

$$i \langle \phi | \dot{\psi} \rangle = \langle \phi | \tilde{H} | \psi \rangle \quad (23)$$

$$i \langle \dot{\phi} | \psi \rangle = - \langle \phi | \tilde{H} | \psi \rangle - \frac{i}{2} \langle \psi | \frac{d}{dt} \left(\frac{\tilde{H} - \langle \tilde{H} \rangle \hat{1}}{\Delta E^2} \right) | \psi \rangle \quad (24)$$

Evaluating the time derivative of the expectation value of the Hamiltonian:

$$\frac{d}{dt} \langle \tilde{H} \rangle = \langle \dot{\psi} | \tilde{H} | \psi \rangle + \langle \psi | \tilde{H} | \dot{\psi} \rangle + \langle \frac{d\tilde{H}}{dt} \rangle \quad (25)$$

$$\frac{d}{dt} \langle \tilde{H} \rangle = \langle \frac{d\tilde{H}}{dt} \rangle \quad (26)$$

This relationship allows us to differentiate under the expectation sign. We now calculate the time derivative of the overlap:

$$\frac{d}{dt} \langle \phi | \psi \rangle = -\frac{1}{2} \langle \frac{d}{dt} \left(\frac{\tilde{H} - \langle \tilde{H} \rangle \hat{1}}{\Delta E^2} \right) \rangle = 0 \quad (27)$$

Therefore we obtain a constant of the motion given by:

$$\langle \phi | \psi \rangle = const. = - \langle \psi | \phi \rangle + \langle \hat{A} \rangle \quad (28)$$

A similar exercise from the variational equation for the Hamiltonian yields:

$$|\phi \rangle = (\langle \psi | \phi \rangle - \frac{(\tilde{H} - \langle \tilde{H} \rangle \hat{1})}{2\Delta E^2} + \hat{G}) |\psi \rangle \quad (29)$$

Directly differentiating this expression with respect to time and substituting the constant of the motion into the equation we derive the fundamental law of motion:

$$(i\frac{d\hat{G}}{dt} + [\hat{G}, \tilde{H}])|\psi\rangle = 0 \quad (30)$$

Some trivial algebra using Hermitian conjugates and right multiplication proves that the G-operator and projection operator follow the Heisenberg equation of motion:

$$i\frac{d\hat{G}}{dt} = [\tilde{H}, \hat{G}] \quad (31)$$

$$i\frac{d\hat{P}}{dt} = [\tilde{H}, \hat{P}] \quad (32)$$

We then use these expressions to evaluate the time derivative of the A-operator by direct differentiation:

$$i\frac{d\hat{A}}{dt} = [\tilde{H}, \hat{A}] + i\hat{P}\frac{d}{dt}\langle \hat{A} \rangle \quad (33)$$

$$\frac{d}{dt}\langle \hat{A} \rangle = 0 \quad (34)$$

$$i\frac{d\hat{A}}{dt} = [\tilde{H}, \hat{A}] \quad (35)$$

It is thus proved that the G-operator, A-operator and projection operator all obey the Heisenberg equation of motion. We shall refer to this important relationship as the Quantum Brachistochrone Equation. This expression forms the foundation of applied time optimal quantum state control and we shall use it repeatedly throughout this paper. As part of our quantum control methodology we will consider an energetic constraint as a Lagrange multiplier in the action functional. This restriction holds the total energy used in the process to be some finite value, and is given by:

$$f_0(\tilde{H}) = Tr(\frac{\tilde{H}^2}{2}) - k = 0 \quad (36)$$

The other constraints which restrict the degrees of freedom for the Hamiltonian of the system will often be linear in \tilde{H} :

$$f_j(\tilde{H}) = Tr(\tilde{H}\hat{g}_j) \quad (37)$$

where the \hat{g}_j 's are traceless Hermitian generators of the unitary space. In this particular instance the Hamiltonian and A-operator will obey the matrix equations:

$$\hat{A} = \lambda_0\tilde{H} + \tilde{F} \quad (38)$$

$$Tr(\tilde{H}\tilde{F}) = 0 \quad (39)$$

$$\tilde{F} = \sum_j \lambda_j \hat{g}_j \quad (40)$$

where the Hamiltonian and associated linear constraint are tracefree Hermitian matrices. We take these results as fundamental for the time optimal evolution of quantum states and may write the equations of motion for our quantum dynamical system as:

$$i \frac{d}{dt} |\Psi(t)\rangle = \tilde{H}[t] |\Psi(t)\rangle \quad (41)$$

$$i \frac{d}{dt} (\tilde{H}[t] + \tilde{F}[t]) = \tilde{H}[t]\tilde{F}[t] - \tilde{F}[t]\tilde{H}[t] \quad (42)$$

$$\text{Tr}(\tilde{H}[t]\tilde{F}[t]) = 0 \quad (43)$$

$$\text{Tr}(\tilde{H}^2[t]/2) = \text{constant} \quad (44)$$

$$\{\hat{G}(t), \hat{P}(t)\} = \hat{G}(t) \quad (45)$$

where $\hat{G} = (\tilde{H} + \tilde{F}) - \left\langle \tilde{H} + \tilde{F} \right\rangle_{\psi}$, and the derivative indicates an explicit differentiation of the matrices with respect to the time parameter. The constraint, boundary condition and Hamiltonian are all Hermitian matrices, so they will evolve unitarily via the Heisenberg equation. When we use co-ordinate transformations, we must use partial derivatives, but the distinction is made clear. Our formalism is explicitly time dependent, in that the matrices derived will be (in general) functions of the time parameter. These are more difficult dynamic problems to solve than the time invariant case, as the matrices within the differential equations to be solved are functions of time instead of constants.

3 Isometry Groups Of Differential Operators And Matrices

In this paper, often we will use isometric transformations to move between various different equivalent physical pictures. Mainly we confine ourselves to the interaction picture or the co-rotating reference frame of Floquet; as such it is quite important that we have a concrete method in place for understanding the meaning and application of isometry and similarity transforms. Firstly, consider the coupled matrix differential equations:

$$i \frac{d}{dt} |\chi(t)\rangle = (\hat{U}^{-1} \tilde{H} \hat{U}) |\chi(t)\rangle \quad (46)$$

$$i \frac{d\hat{U}}{dt} = \left(\sum_k \lambda_k(t) \hat{P}_k \right) \hat{U}(t, 0) \quad (47)$$

We may then write the solution as:

$$|\chi(t)\rangle = \hat{Q}(t,0) |\chi(0)\rangle \quad (48)$$

$$\hat{Q}(t,0) = \exp(-i \int_0^t \hat{U}^{-1}(s,0) \tilde{H}(s) \hat{U}(s,0) ds) \quad (49)$$

$$\hat{U}(s,0) = \exp(-i \int_0^s \sum_k \lambda_k(s') \hat{P}_k ds') \quad (50)$$

Another relevant picture we may use is the Floquet co-rotating reference frame. In this case, we have dynamical equations of motion given by:

$$i \frac{d}{dt} |\varphi(t)\rangle = \left(\frac{\partial \hat{S}}{\partial t} + e^{-i\hat{S}} \tilde{H}(t) e^{+i\hat{S}} \right) |\varphi(t)\rangle \quad (51)$$

which has a solution that can be written in closed form as:

$$|\varphi(t)\rangle = \hat{K}(t,0) |\varphi(0)\rangle \quad (52)$$

$$\hat{K}(t,0) = \exp(-i \int_0^t \hat{W}^{-1}(s,0) \tilde{H}(s) \hat{W}(s,0) ds) \quad (53)$$

$$\hat{W}(s,0) = \exp(-i \int_0^s \frac{\partial \hat{S}}{\partial \tau} d\tau) \quad (54)$$

4 Time Optimal Control For SU(2)

For our first example, we consider the simple SU(2) system. In this case, as the state space is equivalent to a double cover of the Bloch sphere, we expect to recover great circles as our solutions. Our fundamental relations are:

$$\text{Tr}\left(\frac{\tilde{H}^2}{2}\right) - k = 0 \quad (55)$$

$$\text{Tr}(\tilde{H} \hat{\sigma}_z) = 0 \quad (56)$$

$$\text{Tr}(\tilde{H}) = 0 \quad (57)$$

We may write the expression for the Hamiltonian as the ansatz:

$$\tilde{H} = \begin{bmatrix} \alpha & \varepsilon \\ \varepsilon^* & -\alpha \end{bmatrix} \quad (58)$$

Evaluating the trace of the Hamiltonian multiplied with the sigma-z matrix we find directly that $\alpha = 0$. Our time dependent Hamiltonian operator is then:

$$\tilde{H} = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon^* & 0 \end{bmatrix} \quad (59)$$

Computing the quantum brachistochrone and calculating the right hand side of the expression via ordinary matrix multiplication:

$$i \frac{d}{dt} (\tilde{H} + \tilde{F}) = \tilde{H}\tilde{F} - \tilde{F}\tilde{H} \quad (60)$$

we find the explicitly time dependent matrix which defines the optimal control fields to be given by:

$$i \frac{d}{dt} \begin{bmatrix} \Omega & \varepsilon \\ \varepsilon^* & -\Omega \end{bmatrix} = 2\Omega \begin{bmatrix} 0 & -\varepsilon \\ \varepsilon^* & 0 \end{bmatrix} \quad (61)$$

By inspection, $\Omega = \text{const.}$ We may write the differential equations for the complex control fields in the form:

$$i \frac{d}{dt} \begin{bmatrix} \varepsilon(t) \\ \varepsilon^*(t) \end{bmatrix} = 2\Omega \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ \varepsilon^*(t) \end{bmatrix} \quad (62)$$

$$\tilde{H}_{opt}(t) = \begin{bmatrix} 0 & \varepsilon(0)e^{2i\Omega t} \\ \varepsilon^*(0)e^{-2i\Omega t} & 0 \end{bmatrix} \quad (63)$$

At this point the rotating wave approximation would usually be invoked, but as the constraint is a constant of the motion, we may write:

$$(\tilde{H}_{opt}(t) + \Omega\hat{\sigma}_z)\hat{U}(t, 0) = \hat{U}(t, 0)(\tilde{H}_{opt}(0) + \Omega\hat{\sigma}_z) \quad (64)$$

Using the Schrödinger equation, we find the dynamical equation for the unitary:

$$i \frac{d\hat{U}}{dt} = \hat{U}(\tilde{H}_{opt}(0) + \Omega\hat{\sigma}_z) - \Omega\hat{\sigma}_z\hat{U} \quad (65)$$

This expression has an explicit solution:

$$\hat{U}(t, 0) = \exp(i\Omega\hat{\sigma}_z t) \exp(-i(\tilde{H}_{opt}(0) + \Omega\hat{\sigma}_z)t) \quad (66)$$

Some simple algebra using the identity $\exp(i\vec{n}\cdot\hat{\sigma}) = \cos\theta\hat{\mathbf{1}} + i\vec{n}\cdot\hat{\sigma}\sin\theta$ gives the expansion for our unitary over the generators of SU(2) as:

$$\begin{aligned} \hat{U}(t, 0) &= \hat{\mathbf{1}}[\cos(\Omega t) \cos(\Omega' t) + \frac{\Omega}{\Omega'} \sin(\Omega t) \sin(\Omega' t)] \\ &+ \frac{(\varepsilon(0) - \varepsilon^*(0))}{2\Omega'} \sin(\Omega' t) [\hat{\sigma}_x \sin(\Omega t) + \hat{\sigma}_y \cos(\Omega t)] \\ &+ i\hat{\sigma}_z [\cos(\Omega' t) \sin(\Omega t) - \frac{\Omega}{\Omega'} \cos(\Omega t) \sin(\Omega' t)] \end{aligned} \quad (67)$$

where $\Omega' = \sqrt{k + \Omega^2}$. We require a unitary that satisfies boundary conditions of the form:

$$\hat{U}(T, 0)\hat{\sigma}_x\hat{U}^\dagger(T, 0) = -\hat{\sigma}_x \quad (68)$$

which gives us the quantisation condition $\Omega T = m\pi/2$, $m \in \mathbb{N}$. Choosing an initial state $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}[1, 1]^T$ and a final state $|\Psi(T)\rangle = \frac{1}{\sqrt{2}}[1, -1]^T$, we find from the boundary conditions (5) the equivalent conditions on the control fields:

$$\varepsilon(0) = -\varepsilon^*(0) \quad (69)$$

$$\varepsilon(T) = -\varepsilon^*(T) \quad (70)$$

We may rewrite this as:

$$\sin(2\Omega T) = 0 \quad (71)$$

which gives a second quantisation condition:

$$\Omega T = \frac{n\pi}{2}, n \in \mathbb{N} \quad (72)$$

Combining this with our first quantisation condition, we find the period of the wave-vector to be:

$$T^2 = \frac{\pi^2}{4k}(n^2 - m^2) \quad (73)$$

The left hand side of this expression is a positive number, and therefore $n > m$. The minimum time of operation is then:

$$T_{min} = \frac{\pi}{2\sqrt{k}} \quad (74)$$

which is of the form of the Heisenberg energy-time uncertainty principle. We may write the Hamiltonian matrix as:

$$\tilde{H}_{opt}(t) = \nu_0(\sin 2\Omega t \hat{\sigma}_x + \cos 2\Omega t \hat{\sigma}_y) \quad (75)$$

where $\nu_0^2 = k$. Hence the best we can do in terms of energy and time is:

$$\nu_0 T_{min} = \frac{\pi}{2} \quad (76)$$

In summary, we have managed to derive the energy-time relation for a revolving quantum state on $SU(2)$; the arc that is described by our unitary time evolution operator has the appropriate symmetry to be considered a sinusoid convolved with a great circle.

5 Unitary Matrices For $SU(2)$

For completeness, we list a number of useful unitary matrices that appear in the quaternionic geodesic calculation. These are related in various ways through phase angles and transformations; we shall not go into this here as the principal part of the calculation is new work on $SU(3)$. These matrices are:

$$\hat{U}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad (77)$$

$$\hat{U}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\theta} & -e^{-i\theta} \\ e^{i\theta} & e^{-i\theta} \end{bmatrix} \quad (78)$$

$$\hat{U}_3 = i\hat{U}_1\hat{U}_2(\chi) = \begin{bmatrix} \sin \chi & -i \cos \chi \\ i \cos \chi & -\sin \chi \end{bmatrix} \quad (79)$$

$$\hat{U}_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (80)$$

$$\hat{U}_5 = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\alpha} & -e^{-i\alpha} \\ 1 & 1 \end{bmatrix} \quad (81)$$

$$\hat{U}_6 = \exp(-i\vartheta\hat{\sigma}_z) = e^{i\vartheta} \begin{bmatrix} 1 & 0 \\ 0 & e^{-2i\vartheta} \end{bmatrix} \quad (82)$$

$$\hat{U}_j\hat{U}_j^\dagger = \hat{U}_j^\dagger\hat{U}_j = \hat{\mathbf{1}} \quad (83)$$

One may transform various Hamiltonians and density matrices to find interesting relationships between different arrangements of constraints:

$$\hat{H}_P(\varsigma) = \hat{U}_P(\varsigma)\hat{H}(\varsigma)\hat{U}_P^\dagger(\varsigma) \quad (84)$$

$$\frac{1}{2} \begin{bmatrix} e^{-i\alpha} & -e^{-i\alpha} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & e^{-i\alpha} \\ e^{+i\alpha} & 0 \end{bmatrix} \begin{bmatrix} e^{i\alpha} & 1 \\ -e^{i\alpha} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (85)$$

6 Bilinear Quantum Algebra On SU(2)

Writing the quantum brachistochrone in vector form for a single qubit we find:

$$i\frac{d}{dt}(|h(t)\rangle + |f(t)\rangle) = \sum |j\rangle \langle h(t)| \hat{A}_j |f(t)\rangle \quad (86)$$

where the $|j\rangle$'s are the standard basis set, and the matrices \hat{A}_j have the property that $\langle h(t)| \hat{A}_j |f(t)\rangle + \langle f(t)| \hat{A}_j |h(t)\rangle = 0$. Let us calculate the general SU(2) case as an application of the formula. In this situation we have system Hamiltonian and matrix constraints given by:

$$\tilde{H}(t) = \sum_i h_i(t)\hat{\sigma}_i \text{ and } \tilde{F}(t) = \sum_j f_j(t)\hat{\sigma}_j \quad (87)$$

where the Pauli matrices have matrix representation

$$\hat{\sigma}_i \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right\} \quad (88)$$

Our constraints state that the Hamiltonian and $\tilde{F}(t)$ are orthogonal, in that their dot product is zero:

$$\text{Tr}(\tilde{H}(t)\tilde{F}(t)) = \langle h(t)| f(t)\rangle = \langle f(t)| h(t)\rangle = 0 \quad (89)$$

$$\begin{aligned}
i \frac{d}{dt} \begin{bmatrix} (h_z + f_z) & (\varepsilon_h + \varepsilon_f) \\ (\varepsilon_h^* + \varepsilon_f^*) & -(h_z + f_z) \end{bmatrix} \\
= \begin{bmatrix} (\varepsilon_h \varepsilon_f^* - \varepsilon_f \varepsilon_h^*) & +2(h_z \varepsilon_f - f_z \varepsilon_h) \\ -2(h_z \varepsilon_f^* - f_z \varepsilon_h^*) & -(\varepsilon_h \varepsilon_f^* - \varepsilon_f \varepsilon_h^*) \end{bmatrix}
\end{aligned} \tag{90}$$

This may be written in the equivalent vector form as:

$$i \frac{d}{dt} \left(\begin{bmatrix} \sqrt{2} h_z \\ \varepsilon_h \\ \varepsilon_h^* \end{bmatrix} + \begin{bmatrix} \sqrt{2} f_z \\ \varepsilon_f \\ \varepsilon_f^* \end{bmatrix} \right) = \begin{bmatrix} \sqrt{2}(\varepsilon_h \varepsilon_f^* - \varepsilon_f \varepsilon_h^*) \\ 2(h_z \varepsilon_f - f_z \varepsilon_h) \\ -2(h_z \varepsilon_f^* - f_z \varepsilon_h^*) \end{bmatrix} \tag{91}$$

Our Hamiltonian and constraint vectors obey the relation

$$\langle f(t) | \hat{A}_j | h(t) \rangle + \langle h(t) | \hat{A}_j | f(t) \rangle = 0 \tag{92}$$

and hence we finally obtain the control matrices for the vector control system on $SU(2)$ as:

$$\hat{A}_1 = \sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{93}$$

$$\hat{A}_2 = \sqrt{2} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{94}$$

$$\hat{A}_3 = \sqrt{2} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{95}$$

By explicit calculation one can show the identity $\langle h(t) | \hat{A}_j | f(t) \rangle + \langle f(t) | \hat{A}_j | h(t) \rangle = 0$ holds true for all j ; adjoint states are defined on the space $[\mathbb{R}^{n-1}, \mathbb{C}^{k*}, \mathbb{C}^k]$, being the complex conjugate and transposed row vector of the column we started with. Let us now apply this formulation to a well-known example, being the time-dependent Hamiltonian used by Carlini et. al in [1]. The purpose of writing the quantum brachistochrone in vector form is that it enables us to consider the physics directly, which helps with understanding. In this case we have Hamiltonian and constraint matrices as given by:

$$\tilde{H}(t) = \begin{bmatrix} 0 & \varepsilon_h(t) \\ \varepsilon_h^* & 0 \end{bmatrix}; \tilde{F}(t) = \begin{bmatrix} f_z(t) & 0 \\ 0 & -f_z(t) \end{bmatrix} \tag{96}$$

The vectors defined by these matrices may be written as:

$$|h(t)\rangle = \begin{bmatrix} 0 \\ \varepsilon_h(t) \\ \varepsilon_h^*(t) \end{bmatrix}; |f(t)\rangle = \begin{bmatrix} \sqrt{2}f_z(t) \\ 0 \\ 0 \end{bmatrix} \quad (97)$$

where we have used the factor of $\sqrt{2}$ to maintain the trace product relation. These are multivectors in many respects, having a real part and two complex components. The link between the matrix form and the vector control equation is clear; we may use either technique in order to calculate the time optimal Hamiltonian operator.

In this situation we may write the quantum brachistochrone in vector form as:

$$\begin{aligned} i \frac{d}{dt} \begin{bmatrix} \sqrt{2}f_z(t) \\ \varepsilon_h(t) \\ \varepsilon_h^*(t) \end{bmatrix} = & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [\sqrt{2}f_z, 0, 0] \sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon_h \\ \varepsilon_h^* \end{bmatrix} \\ + & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [\sqrt{2}f_z, 0, 0] \sqrt{2} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon_h \\ \varepsilon_h^* \end{bmatrix} \\ + & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [\sqrt{2}f_z, 0, 0] \sqrt{2} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon_h \\ \varepsilon_h^* \end{bmatrix} \\ \therefore i \frac{d}{dt} \begin{bmatrix} \sqrt{2}f_z(t) \\ \varepsilon_h(t) \\ \varepsilon_h^*(t) \end{bmatrix} = & \begin{bmatrix} 0 \\ -2f_z\varepsilon_h \\ +2f_z\varepsilon_h^* \end{bmatrix} \quad (98) \end{aligned}$$

as required. This system may be rewritten in the compact form:

$$\langle h | \hat{D} | h \rangle = \text{const.} = \lambda^2 + |\varepsilon|^2 \quad (99)$$

$$\langle f | \hat{E} | h \rangle = \langle h | \hat{E} | f \rangle = 0 = 2\lambda\Gamma + \varepsilon^*\Pi + \varepsilon\Pi^* \quad (100)$$

$$\langle f | \hat{B}_j | h \rangle = -\langle h | \hat{B}_j | f \rangle \quad (101)$$

$$i \frac{d}{dt} (|h\rangle + |f\rangle) = \sum_j |j\rangle \langle h | \hat{B}_j | f \rangle \quad (102)$$

$$\hat{B}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \hat{B}_2 = 2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad (103)$$

$$\hat{B}_3 = 2 \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (104)$$

$$\hat{D} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \Upsilon & 0 \\ 0 & 0 & (1 - \Upsilon) \end{bmatrix} \quad (105)$$

$$\hat{E} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \quad (106)$$

$$|h\rangle = \begin{bmatrix} \lambda \\ \varepsilon \\ \varepsilon^* \end{bmatrix}, |f\rangle = \begin{bmatrix} \Gamma \\ \Pi \\ \Pi^* \end{bmatrix} \in \begin{bmatrix} \mathbb{R} \\ \mathbb{C}^2 \end{bmatrix} \quad (107)$$

$$\langle h| = [\lambda, \varepsilon^*, \varepsilon], \langle f| = [\Gamma, \Pi^*, \Pi] \in [\mathbb{R}, \mathbb{C}^2] \quad (108)$$

These types of mappings must preserve inner product relationships between the Hamiltonian and constraint.

7 SO(3) Matrix Analysis

Consider a Hamiltonian matrix of the form:

$$\tilde{H}[t] = \vec{n}(t) \cdot \hat{S} = \begin{bmatrix} n_z(t) & 0 & \varepsilon(t) \\ 0 & 0 & 0 \\ \varepsilon^*(t) & 0 & -n_z(t) \end{bmatrix} \quad (109)$$

We form a constraint matrix:

$$\tilde{F}[t] = \begin{bmatrix} u & K_1 & 0 \\ K_1^* & -2u & K_2 \\ 0 & K_2^* & u \end{bmatrix} \quad (110)$$

The quantum brachistochrone equation reads as:

$$i \frac{d}{dt} (\tilde{H}(t) + \tilde{F}(t)) = \tilde{H}(t)\tilde{F}(t) - \tilde{F}(t)\tilde{H}(t) \quad (111)$$

Evaluating this for our matrix system, we find the set of differential equations:

$$\dot{u} = \dot{n}_z = \dot{\varepsilon} = \dot{\varepsilon}^* = 0; \tilde{H}[t] = \tilde{H}[0] \quad (112)$$

Writing our wave-vector in standard form we find the relations:

$$c_2(t) = c_2(0) \quad (113)$$

$$i \frac{d}{dt} \begin{bmatrix} c_1(t) \\ c_3(t) \end{bmatrix} = \begin{bmatrix} n_z & \varepsilon \\ \varepsilon^* & -n_z \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_3(t) \end{bmatrix} \quad (114)$$

$$\text{Tr}(\tilde{H}^2/2) = \text{const.} = n_z^2 + |\varepsilon|^2 = R^2 \quad (115)$$

The propagator may be written in the compact form:

$$\hat{U}(t,0) = \hat{\mathbf{1}} - i \frac{\sin Rt}{R} \tilde{H}[0] + (\cos Rt - 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (116)$$

Choosing an initial state $|\psi(0)\rangle = [1 \ 0 \ 0]^T$ we find the wave-vector:

$$|\psi(t)\rangle = \hat{U}(t,0) |\psi(0)\rangle = \begin{bmatrix} \cos Rt - i \frac{n_z}{R} \sin Rt \\ 0 \\ -\frac{i}{R} \varepsilon^* \sin Rt \end{bmatrix} \quad (117)$$

This is 2π -periodic and satisfies $|\psi(\frac{m\pi}{R})\rangle = |\psi(0)\rangle = |\psi(\frac{2m\pi}{R})\rangle$, where m is an integer.

8 Time Dependent Elliptic Hamiltonian on SU(3)

In this section we shall consider three examples of SU(3) Hamiltonians. The physical motivation for this is to explicitly calculate brachistochrones for some model problems which are more general than the SU(2) models considered in [1]. Boscain et. al has examined various situations on SU(3) that have similar properties, see for example [6]. Our first Hamiltonian matrix is given by:

$$\tilde{H}(t) = \begin{bmatrix} 0 & \alpha(t) & 0 \\ \alpha(t) & 0 & -i\beta(t) \\ 0 & i\beta(t) & 0 \end{bmatrix} \quad (118)$$

with an associated constraint matrix

$$\tilde{F}(t) = \begin{bmatrix} \omega_1(t) & -i\gamma(t) & \varepsilon(t) \\ i\gamma(t) & \omega_2(t) & \kappa(t) \\ \varepsilon^*(t) & \kappa(t) & \omega_3(t) \end{bmatrix}; \sum_{n=1}^3 \omega_j = 0. \quad (119)$$

The quantum brachistochrone equation in vector form then reads as:

$$\frac{d}{dt} \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{bmatrix} = 2 \begin{bmatrix} \gamma\alpha \\ -(\gamma\alpha + \beta\kappa) \\ \beta\kappa \end{bmatrix}; \quad (120)$$

$$\frac{d}{dt} \begin{bmatrix} \gamma(t) \\ \kappa(t) \end{bmatrix} = \begin{bmatrix} -\beta v - (\omega_1 - \omega_3)\alpha \\ -\alpha v + (\omega_2 - \omega_3)\beta \end{bmatrix}; \quad (121)$$

$$\frac{d}{dt} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} = -i\Omega \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \quad (122)$$

$$\frac{d}{dt} \begin{bmatrix} \Omega(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha\kappa + \beta\gamma \end{bmatrix}; \quad (123)$$

where the complex variable $\varepsilon(t) = \Omega(t) - iv(t)$. This has a solution for the time-optimal Hamiltonian which is given by the matrix:

$$\tilde{H}(t) = R \begin{bmatrix} 0 & \cos(\Omega t) & 0 \\ \cos(\Omega t) & 0 & -i\sin(\Omega t) \\ 0 & i\sin(\Omega t) & 0 \end{bmatrix} \quad (124)$$

We now apply this Hamiltonian to a wave-vector, given as a superposition of energy eigenstates which may be (in general) time dependent. This has the expansion:

$$|\Psi(t)\rangle = \Delta_1(t) | +R(t)\rangle + \Delta_2(t) | 0(t)\rangle + \Delta_3(t) | -R(t)\rangle \quad (125)$$

We obtain the formula for the wave-vector in the standard basis $|\Psi(t)\rangle = [c_1(t), c_2(t), c_3(t)]^T$ as given by:

$$c_1(t) = -\frac{\sin(\Omega't)}{\Omega'} \left[i\frac{z_2 R}{\Omega'} + \Omega \left[\frac{z_1}{\Omega'} \cos(\Omega't) - z_3 \sin(\Omega't) \right] \right] + \cos(\Omega t) \left[z_3 \cos(\Omega't) + z_1 \frac{\sin(\Omega't)}{\Omega'} \right] \quad (126)$$

$$c_2(t) = \frac{1}{(\Omega')^2} [\Omega z_2 + iR [z_1 \cos(\Omega't) - z_3 \sin(\Omega't)]] \quad (127)$$

$$c_3(t) = \frac{\cos(\Omega t)}{\Omega'} \left[i\frac{z_2 R}{\Omega'} + \Omega \left[z_1 \frac{\cos(\Omega't)}{\Omega'} - z_3 \sin(\Omega't) \right] \right] - i\sin(\Omega t) \left[z_3 \cos(\Omega't) + \frac{z_1}{\Omega'} \sin(\Omega't) \right] \quad (128)$$

where the parameters z_1, z_2, z_3 are given by the formulae:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \Omega & -iR & 0 \\ -iR & \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta_2(0) \\ \Delta_-(0) \\ \Delta_+(0) \end{bmatrix} \quad (129)$$

$$\Delta_{\pm} = \frac{1}{\sqrt{2}} (\Delta_1 \pm \Delta_3) \quad (130)$$

$$\Omega' = \sqrt{R^2 + \Omega^2} \quad (131)$$

which define our initial state via:

$$|\Psi(0)\rangle = \begin{bmatrix} z_3 \\ \frac{1}{\Omega'^2}(\Omega z_2 + iRz_1) \\ -i \frac{(\Omega z_1 + iRz_2)}{\Omega'^2} \end{bmatrix}; \quad (132)$$

In this case we were able to find a solution despite the explicit time dependence of the system. This example is particularly useful as it indicates a general method of solution for the quantum brachistochrone equation which we may use on the semisimple subgroups.

9 Time Dependent Geodesic on SU(3)

We now take a slightly more general Hamiltonian operator and constraint than the previous example, being the matrices:

$$\tilde{H}(t) = \begin{bmatrix} 0 & \varepsilon_1(t) & 0 \\ \varepsilon_1^*(t) & 0 & \varepsilon_2(t) \\ 0 & \varepsilon_2^*(t) & 0 \end{bmatrix}; \quad (133)$$

$$\tilde{F}(t) = \begin{bmatrix} \omega_1 & 0 & \kappa \\ 0 & \omega_2 & 0 \\ \kappa^* & 0 & -(\omega_1 + \omega_2) \end{bmatrix} \quad (134)$$

Using the quantum brachistochrone equation $i \frac{d}{dt}(\tilde{H} + \tilde{F}) = [\tilde{H}, \tilde{F}]$ we obtain the set of coupled differential equations:

$$\frac{d}{dt} \begin{bmatrix} \kappa(t) \\ \kappa^*(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (135)$$

$$\frac{d}{dt} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (136)$$

$$i \frac{d}{dt} \begin{bmatrix} \varepsilon_1(t) \\ \varepsilon_1^*(t) \\ \varepsilon_2(t) \\ \varepsilon_2^*(t) \end{bmatrix} = \begin{bmatrix} -(\omega_1 - \omega_2)\varepsilon_1 - \kappa\varepsilon_2^* \\ (\omega_1 - \omega_2)\varepsilon_1^* + \kappa^*\varepsilon_2 \\ \kappa\varepsilon_1^* - (\omega_1 + 2\omega_2)\varepsilon_2 \\ -\kappa^*\varepsilon_1 + (\omega_1 + 2\omega_2)\varepsilon_2^* \end{bmatrix} \quad (137)$$

from which we immediately observe that ω_1, ω_2 and κ are constants of the motion. We now assume boundary conditions on the state, such that initially at time zero the wave-vector is in the state $[1, 0, 0]^T$ and it is in the state $[0, 0, 1]^T$ after a time $t = T$. We then have boundary conditions on the projection operator as given by the matrices:

$$\hat{P}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{P}(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (138)$$

From [1], we may write the boundary conditions on the Hamiltonian as:

$$\hat{G}(t) = \left\{ \hat{G}(t), \hat{P}(t) \right\}, 0 \leq t \leq T \quad (139)$$

where

$$\hat{G}(t) = (\tilde{H}(t) + \tilde{F}(t)) - \langle \psi(t) | (\tilde{H}(t) + \tilde{F}(t)) | \psi(t) \rangle \hat{P}(t) \quad (140)$$

After some elementary linear algebra, we find that the boundary conditions are equivalent to a series of relations on the control fields:

$$\omega_1 = \omega_2 = 0 \quad (141)$$

$$\varepsilon_2(0) = \varepsilon_2^*(0) = 0 \quad (142)$$

$$\varepsilon_1^*(T) = \varepsilon_1(T) = 0 \quad (143)$$

As the diagonal elements of $\tilde{F}(t)$ are constant in time, and from the boundary conditions are initially zero, then we can conclude that they are zero throughout the entire motion. This means that the entire constraint matrix is constant for this situation, and we may write the time evolution operator as:

$$\hat{U}(t, 0) = \exp(+it\tilde{F})\exp(-it(\tilde{H}(0) + \tilde{F})) \quad (144)$$

Evaluating these matrices via the eigenvalue determinant, and Laufer's formulae[5] we find an explicit formula for the state as a function of time:

$$|\psi(t)\rangle = \begin{bmatrix} \cos(t\Delta)\cos(|\kappa|t) + \frac{|\kappa|}{\Delta}\sin(|\kappa|t)\sin(t\Delta) \\ -\frac{i\varepsilon_1^*(0)}{\Delta}\sin(t\Delta) \\ i\kappa^* \left[\frac{1}{|\kappa|}\sin(|\kappa|t)\cos(t\Delta) - \frac{1}{\Delta}\sin(t\Delta)\cos(|\kappa|t) \right] \end{bmatrix} \quad (145)$$

where $\Delta = \sqrt{|\kappa|^2 + |\varepsilon_1(0)|^2}$

Calculating the projection operator, we obtain boundary conditions on the state vector:

$$\hat{P}(t) = |\psi(t)\rangle \langle \psi(t)|$$

$$= \begin{bmatrix} |c_1(t)|^2 & c_1(t)c_2^*(t) & c_1(t)c_3^*(t) \\ c_1^*(t)c_2(t) & |c_2(t)|^2 & c_2(t)c_3^*(t) \\ c_1^*(t)c_3(t) & c_2^*(t)c_3(t) & |c_3(t)|^2 \end{bmatrix} \quad (146)$$

After a time $t = T$ the matrix elements of the projection operator satisfy:

$$|c_1(T)|^2 = |c_2(T)|^2 = 0; |c_3(T)|^2 = 1 \quad (147)$$

and hence we may write the equivalent boundary conditions as:

$$|c_2(T)|^2 = 0 = \left(-\frac{i\varepsilon_1^*(0)}{\Delta} \sin(T\Delta)\right)^2 \quad (148)$$

thereby obtaining the condition of time quantisation:

$$\sin(T\Delta) = 0 \Leftrightarrow T = \frac{n\pi}{\sqrt{|\kappa|^2 + |\varepsilon_1(0)|^2}} \quad (149)$$

Considering the boundary conditions on the first parameter from the wavevector, we find immediately that $\cos(T|\kappa|) = 0$ which implies that:

$$T = \frac{(2n' + 1)\pi}{2|\kappa|}, \quad n' \in \mathbb{N} \quad (150)$$

For these to both be the minimum time, they must be equal. From this we find the relationship:

$$|\kappa|^2 \left(1 - \left(\frac{n' + 1/2}{n}\right)^2\right) = \left(\frac{n' + 1/2}{n}\right)^2 |\varepsilon_1(0)|^2 \quad (151)$$

The quantity on the right hand side of the equality is the square of a real number, and hence is positive. This means that the function multiplying $|\kappa|^2$ on the left hand side must be positive, and hence we finally derive the minimum time for this particular state transfer as:

$$n \geq n' + 1/2$$

$$T_{min} = \frac{\pi}{2|\kappa|} = \frac{\sqrt{3}\pi}{2|\varepsilon_1(0)|} \quad (152)$$

After some further algebra we obtain the time-optimal Hamiltonian operator as given by the matrix:

$$\tilde{H}(t) = R \begin{bmatrix} 0 & \cos(kt) & 0 \\ \cos(kt) & 0 & e^{-i\theta} \sin(kt) \\ 0 & e^{+i\theta} \sin(kt) & 0 \end{bmatrix} \quad (153)$$

where $R = |\varepsilon_1(0)|$ and $k = |\kappa|$. Note that as the constraint was calculated to be a constant matrix, we were able to evaluate the time evolution operator without having to calculate the full time-dependent case. All that was required was the time dependence of the constraint, the initial condition of the Hamiltonian operator, and the boundary conditions on the state. This is a very useful technique, as it allows us to move to an alternative reference frame in which for some situations the analysis may be more amenable to solution. Our solution is consistent with the results of [6]. Much of the work that has been conducted on SU(3) Hamiltonians has been related to STIRAP [7], [8], [9]. We do not require their assumption of adiabaticity nor the rotating wave approximation to derive

our results. Our method may be correctly considered to be a competing idea, as it is technically the fastest way to achieve the desired transition as opposed to a slow, steady process as in STIRAP.

10 Triangular Matrices

We now move to the more technical part of this paper, and derive a number of results on matrix groups. The reason for this is to develop a number of representations for angular symmetry on this elliptic group, and also to see if there are any general formulae which may be derived. The intent is to go from simple examples and steadily increase in difficulty in order to solve the main problems contained within. The space of complex matrices has a rich structure, and the dynamics of the semigroups constitutes the behaviour of the time dependent, periodic state. For this reason, we use a one-at-a-time parameter approach, and show how each individual semigroup works.

The simplest way in which a set of square matrices can be divided is into the values above the diagonal, the values below the diagonal, and the diagonal itself. Our space of time dependent Hamiltonian matrix operators have the standard Hermitian property, and this brings us to the consideration of triangular matrices, both upper and lower before we attack the main problem. Form a series of matrices \hat{A} :

$$\hat{A}(a, b, c) = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad (154)$$

We now input some other co-ordinate to form the matrix \hat{A}' :

$$\hat{A}' = \hat{A}(a', b', c') = \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} \quad (155)$$

The structure of the group is given by the anti-commutator and commutator, being the fundamental symmetric and antisymmetric operators:

$$\begin{aligned} \{\hat{A}', \hat{A}\} &= \begin{bmatrix} 1 & 2(a+a') & (b+b') + a'c + c'a \\ 0 & 1 & (b+b') + 2cc' \\ 0 & 0 & 1 \end{bmatrix} \\ &= \hat{A}(a'', b'', c'') \end{aligned} \quad (156)$$

$$[\hat{A}, \hat{A}'] = \hat{A}\hat{A}' - \hat{A}'\hat{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} (b - b') \quad (157)$$

As we can see, these matrices form a nilpotent semigroup:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (158)$$

$$\hat{A}^2(a, b, c) = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad (159)$$

$$= \begin{bmatrix} 1 & 2a & 2b + ca \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{bmatrix} = \hat{A}(2a, 2b + ca, 2c) \quad (160)$$

Useful results:

$$\hat{A}^2 = \hat{A} + \hat{X} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & a & b + ca \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}; [\hat{A}, \hat{X}] = 0 \quad (161)$$

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b + ca \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2a & 2(b + ca) \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{bmatrix} \quad (162)$$

$$\begin{bmatrix} 1 & a & b + ca \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2a & 2(b + ca) \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{bmatrix} \quad (163)$$

The exponential matrix allows us to find the time dependence of the state, and hence solve the differential equations:

$$\exp(-i\hat{A}t) = 1 - i\hat{A}t + \frac{1}{2!}(-i\hat{A}t)^2 + \dots + \frac{1}{n!}(-i\hat{A}t)^n + \dots \quad (164)$$

$$\exp(-i\hat{A}t) = 1 - i\hat{A}t + \frac{1}{2!}(-it)^2(\hat{A} + \hat{X}) \quad (165)$$

$$+ \frac{1}{3!}(-it)^3\hat{A}(\hat{A} + \hat{X}) + \frac{1}{4!}(-it)^4(\hat{A} + \hat{X})^2 + \dots \quad (166)$$

Using the determinant matrix we find the relevant polynomial eigenvalue equation:

$$\left\| \begin{bmatrix} 1 - \lambda & a & b \\ 0 & 1 - \lambda & c \\ 0 & 0 & 1 - \lambda \end{bmatrix} \right\| = \det(\hat{A} - \lambda\hat{1}) = (1 - \lambda)^3 \quad (167)$$

We can find a matrix which under the commutator sends upper triangular to upper triangular:

$$\begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix} = \hat{\Omega} \quad (168)$$

$$\begin{bmatrix} 1 & a(\omega_1 - \omega_2) & b(\omega_1 - \omega_3) \\ 0 & 1 & c(\omega_2 - \omega_3) \\ 0 & 0 & 1 \end{bmatrix} = \hat{A}\hat{\Omega} - \hat{\Omega}\hat{A} \quad (169)$$

We observe the determinant allows us to invert the system of equations:

$$\det(\hat{A}) = \begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad (170)$$

Writing the differential equations in equivalent matrix form:

$$\hat{A}\vec{x} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + ax_2 + bx_3 \\ x_2 + cx_3 \\ x_3 \end{bmatrix} \quad (171)$$

$$\hat{A}\vec{x} = \frac{d\vec{x}}{dt} \leftrightarrow \begin{bmatrix} x_1 + ax_2 + bx_3 \\ x_2 + cx_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{bmatrix} \quad (172)$$

We solve the system:

$$\frac{d^n x_1}{dt^n} = x_1 + nax_2 + (nb + ac)x_3 \quad (173)$$

$$x_1(t) = \left(\frac{ac}{2}t^2 + (bc + ak)t + \varrho \right) e^t \quad (174)$$

$$x_2(t) = (k + ct)e^t \quad (175)$$

$$x_3(t) = ce^t \quad (176)$$

The dimensionality of the matrix group we are after, being the hermitian operators can be broken down into a sum of triangular arrays. The size of the dimension of the space is given by:

$$1 + 2(1) = 2^2 - 1 \quad (177)$$

$$2 + 2(1 + 2) = 3^2 - 1 \quad (178)$$

$$3 + 2(1 + 2 + 3) = 4^2 - 1 \quad (179)$$

$$n^2 = n + 2 \sum_{j=1}^{n-1} j \quad (180)$$

Taking a lower triangular matrix, and evaluating the commutator with an upper dimensional matrix, we find

$$\hat{B} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \quad (181)$$

$$[\hat{A}, \hat{B}] = \begin{bmatrix} -(ax + by) & -bz & 0 \\ -cy & ax - cz & bx \\ 0 & ay & by + cz \end{bmatrix} \quad (182)$$

Let

$$x = a^*, y = b^*, z = c^*$$

then we find:

$$[\hat{A}, \hat{B}] = \begin{bmatrix} -(|a|^2 + |b|^2) & -bc^* & 0 \\ -cb^* & |a|^2 - |c|^2 & ba^* \\ 0 & b^*a & |b|^2 + |c|^2 \end{bmatrix} \quad (183)$$

Other than the diagonal entries, this matrix is close to the required symmetry of the elliptic matrix operators. The relationship between triangular matrices and Hermitian matrices is quite useful for understanding many of the problems we deal with. A Hermitian matrix may be given as a sum of an upper triangular, its complex conjugate transpose, and a diagonal matrix. We now move to consideration of the particular Hamiltonian matrix.

$$\tilde{H}(t) = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & -i\beta \\ 0 & i\beta & 0 \end{bmatrix} = \tilde{H} \quad (184)$$

The Cayley-Hamilton formula for this matrix reads as:

$$\tilde{H}^3 = (\alpha^2 + \beta^2)\tilde{H} \quad (185)$$

and we can therefore write the propagator in the form:

$$\exp(-i\theta\tilde{H}) = 1 + g_1(\theta)\tilde{H} + g_2(\theta)\tilde{H}^2 \quad (186)$$

$$\left\| \begin{array}{ccc} -\lambda & \alpha & 0 \\ \alpha & -\lambda & -i\beta \\ 0 & i\beta & -\lambda \end{array} \right\| = -\lambda^3 + \lambda(\alpha^2 + \beta^2) \quad (187)$$

$$\left[\tilde{H}, \frac{d\tilde{H}}{dt} \right] = i(\dot{\alpha}\beta - \alpha\dot{\beta}) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (188)$$

$$\left[\frac{d\tilde{H}}{dt}, \tilde{H}^2 \right] = (\alpha^2 + \beta^2) \frac{d\tilde{H}}{dt} + (\alpha\dot{\alpha} + \beta\dot{\beta}) \begin{bmatrix} 0 & +\alpha & 0 \\ -\alpha & 0 & i\beta \\ 0 & -i\beta & 0 \end{bmatrix} \quad (189)$$

$$\alpha\dot{\alpha} + \beta\dot{\beta} = 0 \Rightarrow \left[\frac{d\tilde{H}}{dt}, \tilde{H}^2 \right] = C \frac{d\tilde{H}}{dt} \quad (190)$$

$$\tilde{H}(T) = \tilde{H}(0) \Rightarrow \tilde{H}(T) = \hat{U}^\dagger(T, 0)\tilde{H}(0)\hat{U}(T, 0) \quad (191)$$

$$\left[\tilde{H}(T), \hat{U}(T, 0) \right] = \left[\tilde{H}(0), \hat{U}(T, 0) \right] = 0 \quad (192)$$

A periodic system has $\tilde{H}(nT) = \tilde{H}(0)$ for $n \in \mathbb{N}$, so we obtain:

$$\left[\tilde{H}(0), \hat{U}(nT, 0) \right] = 0 \quad (193)$$

$$\hat{U}(nT, 0) = e^{-in\omega T} \hat{\mathbf{1}} \quad (194)$$

Initially, the time evolution operator has value unity:

$$\hat{U}(0, 0) = \hat{\mathbf{1}} \quad (195)$$

$$i \frac{d\hat{U}}{dt} = \tilde{H}(t)\hat{U}(t, 0) \quad (196)$$

$$\tilde{H}(t) = R \begin{bmatrix} 0 & \cos(\omega t) & 0 \\ \cos(\omega t) & 0 & -i\sin(\omega t) \\ 0 & i\sin(\omega t) & 0 \end{bmatrix} \quad (197)$$

This operator is periodic:

$$\tilde{H}\left[\frac{\pi}{\omega}(2n+1)\right] = \tilde{H}[0] \quad (198)$$

with initial value:

$$\tilde{H}[0] = \begin{bmatrix} 0 & R & 0 \\ R & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (199)$$

Time commuting identities describe the actions of the subgroup:

$$\left[\tilde{H}(t), \tilde{H}(s) \right] = i(\beta(t)\alpha(s) - \beta(s)\alpha(t)) \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad (200)$$

We may directly integrate the Hamiltonian matrix to obtain:

$$\begin{aligned}
\int_t^{t+T} \tilde{H}[s] ds = & \\
& \frac{R}{\omega} \sin(\omega t) \begin{bmatrix} 0 & \cos(\omega T) - 1 & 0 \\ \cos(\omega T) - 1 & 0 & -i \sin(\omega T) \\ 0 & i \sin(\omega T) & 0 \end{bmatrix} \\
& + \frac{R}{\omega} \cos(\omega t) \begin{bmatrix} 0 & \sin(\omega T) & 0 \\ \sin(\omega T) & 0 & i(\cos(\omega T) - 1) \\ 0 & -i(\cos(\omega T) - 1) & 0 \end{bmatrix} \quad (201)
\end{aligned}$$

To calculate the full time dependence of the wave-vector, we must exponentiate the above equation, which seems daunting. However, there are useful tricks that may be applied consistently to bring the time evolution operator into a more useful and physically intuitive format.

11 Invariants Of SU(3)

The invariants of a system, in a sense, define the system itself. Several combinations of the components of the wave-vector arise commonly during calculations on SU(3), so by considering dynamical perspectives from this transported reference frame we are able to see the crux of the physics.

$$\tilde{H}(t) = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & -i\beta \\ 0 & i\beta & 0 \end{bmatrix} \quad (202)$$

Redefining our system of wave-vector co-ordinates and constraints:

$$f_1 = c_1 c_3^* - c_1^* c_3; \quad f_2 = c_2 c_3^* - c_2^* c_3; \quad f_3 = c_1 c_2^* + c_1^* c_2;$$

$$f_4 = c_2 c_3^* + c_2^* c_3; \quad f_5 = c_1 c_3^* + c_1^* c_3; \quad f_6 = c_1 c_2^* - c_1^* c_2;$$

$$|c_1|^2 + |c_2|^2 + |c_3|^2 = 1; \quad \alpha^2 + \beta^2 = \text{const.} \quad (203)$$

We immediately find the system of equations:

$$\Re(f_{1,2}) = 0 \quad (204)$$

$$\Im(f_3) = 0 \quad (205)$$

This means the domains of these functions are constrained in the complex plane. We also find in general that:

$$\frac{d}{dt} \langle \tilde{H} \rangle = \left\langle \frac{d\tilde{H}}{dt} \right\rangle \quad (206)$$

Substituting our transformed variables into the original dynamical equations we obtain the system:

$$\frac{df_1}{dt} = \frac{\alpha}{i} f_4 - \beta f_3; \quad \frac{df_2}{dt} = \frac{\alpha}{i} f_5 - 2\beta |c_2|^2; \quad (207)$$

$$\frac{df_3}{dt} = -\beta f_5; \quad \frac{df_4}{dt} = \frac{\alpha}{i} f_1 - 2\beta |c_3|^2 \quad (208)$$

$$\frac{df_5}{dt} = \frac{\alpha}{i} f_2 - \beta f_6; \quad (209)$$

$$\frac{df_6}{dt} = 2\alpha(|c_2|^2 - |c_1|^2) - \beta f_2; \quad (210)$$

$$\alpha \frac{df_3}{dt} + i\beta \frac{df_2}{dt} = 0 \quad (211)$$

Defining an observable \hat{O} and our algebra of states:

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle \quad (212)$$

$$|\psi\rangle = \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{bmatrix}, \langle \psi | = [c_1^* \quad c_2^* \quad c_3^*] = (|\psi\rangle^*)^T \quad (213)$$

we derive the useful relations:

$$\frac{d}{dt} \langle \hat{O} \rangle = \left\langle \frac{d\hat{O}}{dt} \right\rangle + i \langle [\tilde{H}, \hat{O}] \rangle \quad (214)$$

$$\frac{d}{dt} \langle \hat{O}^2 \rangle = \left\langle \tilde{O}, \frac{d\hat{O}}{dt} \right\rangle \quad (215)$$

After some work, we obtain the fundamental dynamics:

$$\begin{aligned} \left\langle \left\{ \tilde{H}, \frac{d\tilde{H}}{dt} \right\} \right\rangle &= i(\alpha\dot{\beta} + \dot{\alpha}\beta) f_1 \\ &+ 2(\alpha\dot{\alpha} |c_1|^2 + (\alpha\dot{\alpha} + \dot{\beta}\beta) |c_2|^2 + \beta\dot{\beta} |c_3|^2) \end{aligned} \quad (216)$$

$$[\tilde{H}, \tilde{H}^2] = 0 \quad (217)$$

$$\frac{d}{dt} \langle \tilde{H}^2 \rangle = \left\langle \left\{ \tilde{H}, \frac{d\tilde{H}}{dt} \right\} \right\rangle \quad (218)$$

The average of the energy, energy variance and energy squared may be given in terms of these functions for this system.

$$\Delta E = \sqrt{\langle \dot{\psi} | (\mathbf{1} - |\psi\rangle\langle\psi|) | \dot{\psi} \rangle} \quad (219)$$

$$\langle \tilde{H} \rangle = \alpha(c_1^*c_2 + c_1c_2^*) + i\beta(c_2c_3^* - c_2^*c_3) \quad (220)$$

$$\begin{aligned} \langle \tilde{H}^2 \rangle &= \alpha^2(|c_1|^2 + |c_2|^2) + \beta^2(|c_2|^2 + |c_3|^2) \\ &\quad + i\beta\alpha(c_1c_3^* - c_1^*c_3) \end{aligned} \quad (221)$$

The total amount of probability is conserved:

$$\frac{d}{dt} \langle \psi(t) | \psi(t) \rangle = 0 \quad (222)$$

hence we find:

$$\beta^2 \frac{d}{dt} |c_1|^2 + \alpha^2 \frac{d}{dt} |c_3|^2 = i\beta\alpha \frac{d}{dt} (c_1c_3^* - c_1^*c_3) \quad (223)$$

$$\alpha\beta(\alpha c_2^*c_3 + \beta c_1^*c_2) = 0 \quad (224)$$

$$\alpha^2 + \beta^2 = R^2 \quad \Rightarrow \quad \alpha\dot{\alpha} + \beta\dot{\beta} = 0 \quad (225)$$

$$\frac{d}{dt} (|c_1|^2 + |c_2|^2 + |c_3|^2) = 0 \quad \Rightarrow \quad \frac{\alpha}{i} f_3 + \beta f_4 = 0 \quad (226)$$

To calculate the probability of any particular event, we use:

$$P(j|\psi) = |c_j(t)|^2 = |\langle j|\psi \rangle|^2 \quad (227)$$

Obviously this is exhaustive and sums to unity.

12 Quantum Frenet-Serre Formula

The Frenet-Serre formula of classical differential geometry relates the path of a particle to various properties of the surface it is rolling along, being the torsion and the curvature. This equation has a fundamental similarity to our Hamiltonian matrix, and as such we consider a new example, which is equivalent in all respects to the Frenet-Serre formula. By applying our method to a problem with known solution, we are providing a valuable check to our formalism. Firstly, assume the dynamical system:

$$i \frac{d}{dt} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & -iK(t) & 0 \\ iK(t) & 0 & -iT(t) \\ 0 & iT(t) & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad (228)$$

Writing the explicitly parameter dependent Hamiltonian matrix and constraint:

$$\tilde{H}(t) = \begin{bmatrix} 0 & -iK(t) & 0 \\ iK(t) & 0 & -iT(t) \\ 0 & iT(t) & 0 \end{bmatrix}; \tilde{F} = \begin{bmatrix} \omega_1 & \alpha & \chi \\ \alpha & \omega_2 & \beta \\ \chi^* & \beta & \omega_3 \end{bmatrix} \quad (229)$$

$$K, T, \alpha, \beta, \omega_j \in f : [0, \infty) \rightarrow \mathbb{R} \quad (230)$$

$$\chi, \chi^* \in f : [0, \infty) \rightarrow \mathbb{C} \quad (231)$$

$$\omega_1 + \omega_2 + \omega_3 = 0 \quad (232)$$

Evaluating the quantum brachistochrone equation:

$$i \frac{d}{dt} (\tilde{H} + \tilde{F}) = [\tilde{H}, \tilde{F}] \quad (233)$$

we obtain the equivalent set of relations:

$$\Im(\chi) = \text{const.} = \eta \quad (234)$$

$$\frac{d}{dt} \left(\frac{\chi - \chi^*}{2i} \right) = 0 \quad (235)$$

Hence we may write the dynamics of the curve in matrix form as:

$$\frac{d}{dt} \begin{bmatrix} K(t) \\ T(t) \end{bmatrix} = \begin{bmatrix} 0 & -\eta \\ \eta & 0 \end{bmatrix} \begin{bmatrix} K(t) \\ T(t) \end{bmatrix} \quad (236)$$

This admits solutions:

$$T(t) = A \sin(\eta t) + B \cos(\eta t) \quad (237)$$

$$K(t) = C \sin(\eta t) + N \cos(\eta t) \quad (238)$$

There is no real difference between our method and the Frenet-Serre method other than where (and if) you take the complex unit:

$$\frac{d|\psi\rangle}{dt} = \hat{A}(t) |\psi\rangle \quad (239)$$

$$\tilde{H}(t) = i\hat{A}(t) \quad (240)$$

The solution Hamiltonian has periodicity and initial values given by:

$$\tilde{H}(0) = \tilde{H}\left(\frac{2n\pi}{\eta}\right) = i \begin{bmatrix} 0 & -N & 0 \\ N & 0 & -B \\ 0 & B & 0 \end{bmatrix} \quad (241)$$

$$\tilde{H}\left(\frac{n\pi}{\eta}\right) = (-1)^n i \begin{bmatrix} 0 & -C & 0 \\ C & 0 & -A \\ 0 & A & 0 \end{bmatrix} \quad (242)$$

In terms of the unitary this means we can write:

$$\hat{U}(T, 0) \tilde{H}(0) \hat{U}^\dagger(T, 0) = \tilde{H}(T) = \tilde{H}(0) \quad (243)$$

Our eigenvalue equation reads as:

$$\det(\tilde{H} - \lambda \hat{\mathbf{1}}) = \begin{vmatrix} -\lambda & -iK & 0 \\ iK & -\lambda & -iT \\ 0 & iT & -\lambda \end{vmatrix} = 0 \quad (244)$$

Giving us the cubic polynomial:

$$-\lambda^3 + \lambda(K^2 + T^2) = 0 \quad (245)$$

Now, our initial Hamiltonian was constrained to finite energy:

$$\text{Tr}(\tilde{H}^2/2) = \text{const.} \quad (246)$$

Hence:

$$K^2 + T^2 = R^2 \quad (247)$$

The solution is a circle. This fact is well known. One may consider various different scenarios with respect to the Hamiltonian-time symmetry:

$$\langle \psi(0) | \tilde{H}(t) | \psi(0) \rangle = \langle \psi(t) | \tilde{H}(0) | \psi(t) \rangle \quad (248)$$

Computing this for our Hamiltonian matrix we find:

$$\langle \psi(t) | \tilde{H}(0) | \psi(t) \rangle = iR [e^{i\omega t} z - e^{-i\omega t} z^*] \quad (249)$$

$$\frac{d}{dt} \left\{ \langle \psi(0) | \tilde{H}(t) | \psi(0) \rangle \right\} = \frac{1}{i} \langle \psi(t) | [\tilde{H}(0), \tilde{H}(t)] | \psi(t) \rangle \quad (250)$$

If we rescale the time parameter, the Hamiltonian takes the simpler form:

$$\tilde{H}(t) \Rightarrow \begin{bmatrix} 0 & -i\cos(t) & 0 \\ i\cos(t) & 0 & -i\sin(t) \\ 0 & i\sin(t) & 0 \end{bmatrix}, t \Rightarrow Rt \quad (251)$$

The dynamical equation of state is:

$$\tilde{H} |\psi\rangle = R(C_+ |+\rangle - C_- |-\rangle) \quad (252)$$

which has eigenvectors given by:

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -i\cos(t) \\ 1 \\ i\sin(t) \end{bmatrix} \quad (253)$$

$$|0\rangle = \begin{bmatrix} i\sin(t) \\ 0 \\ i\cos(t) \end{bmatrix} \quad (254)$$

$$|-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i\cos(t) \\ 1 \\ -i\sin(t) \end{bmatrix} \quad (255)$$

It is pleasing that this methodology delivers the standard answer to such a well-known question. The total curvature is a constant; the geometric figure is periodic.

13 Unitary Transformations of SU(3)

We wish to find all the unitary transformations of SU(3), in order to understand the geometry of states. In this case, our first step is to examine the transformations that take the time dependent Hamiltonian matrix from the previous worked example to a diagonal representation:

$$\tilde{H}_{FS}(t) = \hat{D}(t)\hat{L}\hat{D}^\dagger(t) \quad (256)$$

$$\hat{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (257)$$

We find:

$$\hat{D}(t) = \begin{bmatrix} -\frac{i}{\sqrt{2}}\cos(t) & \frac{i}{\sqrt{2}}\cos(t) & i\sin(t) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}}\sin(t) & -\frac{i}{\sqrt{2}}\sin(t) & i\cos(t) \end{bmatrix} \quad (258)$$

This has the required unitary properties:

$$\hat{D}(t)\hat{D}^\dagger(t) = \hat{D}^\dagger(t)\hat{D}(t) = \hat{\mathbf{1}} \quad (259)$$

with initial value given by:

$$\hat{D}(0) = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix} \quad (260)$$

We may write the time evolution of this operator in terms of an integral equation:

$$\hat{D}(t) = \exp(i\hat{L}t)\hat{D}(0) - i \int_0^t \tilde{H}(s)\hat{D}(s)ds \quad (261)$$

Now, consider the eigenvectors of our elliptic Hamiltonian matrix:

$$\begin{bmatrix} 0 & \cos\phi & 0 \\ \cos\phi & 0 & -i\sin\phi \\ 0 & i\sin\phi & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \cos\phi \\ 1 \\ i\sin\phi \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos\phi \\ 1 \\ i\sin\phi \end{bmatrix} \quad (262)$$

$$\begin{bmatrix} 0 & \cos\phi & 0 \\ \cos\phi & 0 & -i\sin\phi \\ 0 & i\sin\phi & 0 \end{bmatrix} \begin{bmatrix} -\sin\phi \\ 0 \\ i\cos\phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (263)$$

$$\begin{bmatrix} 0 & \cos\phi & 0 \\ \cos\phi & 0 & -i\sin\phi \\ 0 & i\sin\phi & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -\cos\phi \\ 1 \\ -i\sin\phi \end{bmatrix} = \frac{-1}{\sqrt{2}} \begin{bmatrix} -\cos\phi \\ 1 \\ -i\sin\phi \end{bmatrix} \quad (264)$$

Writing a matrix with these eigenvectors as columns, we obtain:

$$\hat{J}(\phi) = \begin{bmatrix} \frac{1}{\sqrt{2}}\cos\phi & -\frac{1}{\sqrt{2}}\cos\phi & -\sin\phi \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}}\sin\phi & -\frac{i}{\sqrt{2}}\sin\phi & i\cos\phi \end{bmatrix} \quad (265)$$

This matrix is unitary:

$$\hat{J}(\phi)\hat{J}^\dagger(\phi) = \hat{J}^\dagger(\phi)\hat{J}(\phi) = \hat{\mathbf{1}} \quad (266)$$

$$\hat{J}(\phi)\hat{L}\hat{J}^\dagger(\phi) = \begin{bmatrix} 0 & \cos\phi & 0 \\ \cos\phi & 0 & -i\sin\phi \\ 0 & i\sin\phi & 0 \end{bmatrix} \quad (267)$$

$$\hat{J}(0) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & i \end{bmatrix} \quad (268)$$

and has initial value given by a Hadamard gate with a $\frac{\pi}{2}$ rotation on the ancilla. Finally, using the completely complex elliptic Hamiltonian

$$\tilde{H} = \begin{bmatrix} 0 & \epsilon_1 & 0 \\ \epsilon_1^* & 0 & \epsilon_2 \\ 0 & \epsilon_2^* & 0 \end{bmatrix} \quad (269)$$

Eigenvectors:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} \cos\varphi \\ 1 \\ ie^{i\varrho}\sin\varphi \end{bmatrix} \quad \& \quad \begin{bmatrix} ie^{-i\varrho}\sin\varphi \\ 0 \\ \cos\varphi \end{bmatrix} \quad (270)$$

$$\& \frac{1}{\sqrt{2}} \begin{bmatrix} -\cos\varphi \\ 1 \\ -ie^{i\varrho}\sin\varphi \end{bmatrix}, \quad \varrho = \text{const.} \in [0, 2\pi)$$

$$\hat{Q}(\varphi) = \begin{bmatrix} \frac{1}{\sqrt{2}}\cos\varphi & -\frac{1}{\sqrt{2}}\cos\varphi & ie^{-i\varrho}\sin\varphi \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}}e^{i\varrho}\sin\varphi & -\frac{i}{\sqrt{2}}e^{i\varrho}\sin\varphi & \cos\varphi \end{bmatrix} \quad (271)$$

$$\hat{Q}(0) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (272)$$

The initial value of this operator is a Hadamard matrix with an ancilla bit set to 1. It is unitary:

$$\hat{Q}^\dagger(\varphi)\hat{Q}(\varphi) = \hat{Q}(\varphi)\hat{Q}^\dagger(\varphi) = 1 \quad (273)$$

$$\hat{Q}(\varphi)\hat{L}\hat{Q}^\dagger(\varphi) = \begin{bmatrix} 0 & \cos\varphi & 0 \\ \cos\varphi & 0 & -ie^{-i\varrho}\sin\varphi \\ 0 & ie^{i\varrho}\sin\varphi & 0 \end{bmatrix} \quad (274)$$

We may summarise these formulae on unitaries into the compact identities:

$$S(\hat{K}[t]; t) = \int_0^t \hat{K}(s)\hat{L}\hat{K}^\dagger(s)ds \quad (275)$$

$$\frac{\delta S}{\delta \Lambda} = 0 \Rightarrow \frac{\delta \hat{K}}{\delta \Lambda}\hat{L}\hat{K}^\dagger + \hat{K}\hat{L}\frac{\delta \hat{K}^\dagger}{\delta \Lambda} = 0 \quad (276)$$

$$[\hat{H}, \hat{K}] \hat{L}\hat{K}^\dagger = \hat{K}\hat{L} [\hat{H}, \hat{K}^\dagger] \quad (277)$$

$$\{\hat{H}, \hat{K}\hat{L}\hat{K}^\dagger\} = \hat{K}\{\hat{H}, \hat{L}\}\hat{K}^\dagger \quad (278)$$

Our propagator for the full Hamiltonian may be written in the compact form:

$$\begin{aligned}\hat{U}(\theta, 0) &= \exp\left(-i \int_0^\theta ds \tilde{H}(s)\right) \\ &= \begin{bmatrix} 1 + (c-1)c^2 & -isc & -i(c-1)sc \\ -isc & c & -s^2 \\ i(c-1)sc & s^2 & 1 + (c-1)s^2 \end{bmatrix} \\ & \quad c = \cos\theta, \quad s = \sin\theta\end{aligned}\quad (279)$$

$$\hat{U}^\dagger(\theta, 0)\hat{U}(\theta, 0) = \hat{U}(\theta, 0)\hat{U}^\dagger(\theta, 0) = \hat{\mathbf{1}} \quad (280)$$

$$\hat{U}(\theta, 0) = \hat{\mathbf{1}} - isin\theta\tilde{H} + (\cos\theta - 1)\tilde{H}^2 \quad (281)$$

One final useful unitary transformation is the equivalent NOT-gate, which can be used to flip two state labels:

$$\hat{N} \begin{bmatrix} 0 & \epsilon_1 & 0 \\ \epsilon_1^* & 0 & \epsilon_2 \\ 0 & \epsilon_2^* & 0 \end{bmatrix} \hat{N}^\dagger = \begin{bmatrix} 0 & \epsilon_2^* & 0 \\ \epsilon_2 & 0 & \epsilon_1^* \\ 0 & \epsilon_1 & 0 \end{bmatrix}; \hat{N} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (282)$$

Writing $|v_j\rangle$ for the j -th column of \hat{U} , it is simple to show that $\langle v_j | v_k \rangle = \delta_{jk}$. Any rotation in this space is composed of these fundamental operators. We now compute commutators of our fundamental transformations with the matrix \hat{L} :

$$\begin{aligned}\left[\hat{L}, \hat{U}(\theta, 0)\right] &= \\ & \begin{bmatrix} 0 & -2isin\theta\cos\theta & i(\cos\theta - 1)\sin\theta\cos\theta \\ -2isin\theta\cos\theta & 0 & \sin^2\theta \\ -isin\theta\cos\theta(\cos\theta - 1) & -\sin^2\theta & 0 \end{bmatrix}\end{aligned}\quad (283)$$

$$\left[\hat{L}, \hat{J}(\phi)\right] = \begin{bmatrix} 0 & -\sqrt{2}\cos\phi & -\sin\phi \\ -\sqrt{2} & 0 & 0 \\ -\frac{i}{\sqrt{2}}\sin\phi & -\frac{i}{\sqrt{2}}\sin\phi & 0 \end{bmatrix} \quad (284)$$

$$\left[\hat{L}, \hat{Q}(\varphi)\right] = \begin{bmatrix} 0 & -\sqrt{2}\cos\varphi & ie^{-i\varphi}\sin\varphi \\ -\sqrt{2} & 0 & 0 \\ -\frac{i}{\sqrt{2}}e^{i\varphi}\sin\varphi & -\frac{i}{\sqrt{2}}e^{i\varphi}\sin\varphi & 0 \end{bmatrix} \quad (285)$$

$$\left[\hat{L}, \hat{D}(\chi)\right] = \begin{bmatrix} 0 & i\sqrt{2}\cos\chi & i\sin\chi \\ -\sqrt{2} & 0 & 0 \\ -\frac{i}{\sqrt{2}}\sin\chi & -\frac{i}{\sqrt{2}}\sin\chi & 0 \end{bmatrix} \quad (286)$$

Hence our fundamental rotations form compact semi-groups with the operator \hat{L} . One may define a sequence of polynomials via the time evolution operator, viz.:

$$\hat{U}^{-1}(\theta, 0) = \left[\frac{P_{l,m}(\cos\theta)}{q(\cos\theta)} \right] \quad (287)$$

We have eigenreflections:

$$\hat{M}_j = \hat{\mathbf{1}} - |v_k\rangle \langle v_k| \quad (288)$$

$$\hat{M}_1 = \frac{1}{2} \begin{bmatrix} 1 + \sin^2\xi & -\cos\xi & i\sin\xi\cos\xi \\ -\cos\xi & 1 & i\sin\xi \\ -i\sin\xi\cos\xi & -i\sin\xi & 1 + \cos^2\xi \end{bmatrix} \quad (289)$$

$$\hat{M}_2 = \begin{bmatrix} \sin^2\xi & 0 & -i\sin\xi\cos\xi \\ 0 & 1 & 0 \\ i\sin\xi\cos\xi & 0 & 1 + \cos^2\xi \end{bmatrix} \quad (290)$$

$$\hat{M}_3 = \frac{1}{2} \begin{bmatrix} 1 + \sin^2\xi & \cos\xi & i\sin\xi\cos\xi \\ \cos\xi & 1 & -i\sin\xi \\ -i\sin\xi\cos\xi & i\sin\xi & 1 + \cos^2\xi \end{bmatrix} \quad (291)$$

These eigenreflections obey the periodicity condition:

$$\hat{M}_2(-\xi) = \hat{M}_1(\xi) + \hat{M}_3(\xi) \quad (292)$$

which is directly related to the equation of state:

$$|3\rangle_{(t+\frac{\pi}{2})} - |1\rangle_{(t+\frac{\pi}{2})} = \sqrt{2} |2\rangle_{(t)} \quad (293)$$

14 Elliptic Polynomials

Using the formula for the propagator, and making the substitution $z = \cos\theta$ we obtain the associated z-polynomials:

$$P_{l,m}(z) = q(z) \left[\hat{U}^{-1} \right]_{l,m} \quad (294)$$

We display several of these functions:

$$q(z) = 4z^7 - 4z^6 - 8z^5 + 8z^4 + 4z^3 - 6z^2 + 1 \quad (295)$$

$$P_{1,1}(z) = 2z^4 - z^3 - 3z^2 + 1 = p(z) \quad (296)$$

$$P_{3,3}(z) = -2z^4 + z^3 + z^2 - z = P(z) \quad (297)$$

$$q(z) + P_{3,3}(z) = Q(z) \quad (298)$$

$$Q(z) = 4z^7 - 4z^6 - 8z^5 + 6z^4 + 5z^3 - 5z^2 - z + 1 \quad (299)$$

We may Laplace z-transform:

$$L[F(z)] = L_F[s] = \int_0^\infty F[z(\theta)]e^{-s\theta} d\theta \quad (300)$$

These transformations enable us to move between the different representations in this space. They are analogous to the matrices used earlier to transform to the eigenbasis. These transformed functionals are rational polynomials with various interesting properties:

$$L_q[s] = \frac{b_q(s)}{r_q(s)}; \quad (301)$$

$$L_Q[s] = \frac{b_Q(s)}{r_Q(s)}; \quad L_P[s] = \frac{b_P(s)}{r_P(s)} \quad (302)$$

$$b_q(s) = s^7 - 125s^5 + 24s^4 + 192s^3 - 960s^2 - 2880s + 20160 \quad (303)$$

$$r_q(s) = s^8 \quad (304)$$

$$b_Q(s) = -2(s^{12} + 120s^{10} + 5016s^8 + 86527s^6 + 550413s^4 + 895923s^2 + 396900) \quad (305)$$

$$r_Q = s(s^2 + 1^2)(s^2 + 2^2) \times \cdots \times (s^2 + 7^2) = s \prod_{k=1}^7 (s^2 + k^2) \quad (306)$$

$$b_P(s) = -(s^8 + 29s^6 + 208s^4 + 306s^2 - 144) \quad (307)$$

$$r_P(s) = r_P(s) = s \prod_{k=1}^4 (s^2 + k^2) \quad (308)$$

For a small circle centred on the origin:

$$\oint_{\Gamma} \frac{b_q(z)}{r_q(z)} dz = 1 \quad \oint_{\Gamma} \frac{b_P(z)}{r_P(z)} dz = \frac{1}{4} \quad (309)$$

$$\oint_{\Gamma} \frac{b_Q(z)}{r_Q(z)} dz = -\frac{1}{2} \quad \oint_{\Gamma} \frac{b_P(z)}{r_P(z)} dz = -\frac{1}{4} \quad (310)$$

We may further Fourier transform our functions to find e.g.:

$$F_W(k) = \int_0^\infty e^{-iks} L_W(s) ds \quad (311)$$

$$F_P(k) = \int_0^\infty e^{-iks} L_P(s) ds \quad (312)$$

$$F_P(k) = \frac{i\pi}{4} (u[-k]H[k] - u[k]H[-k]) \quad (313)$$

$$u[k] = e^{4k} - e^{3k} + 2e^{2k} + e^k + 1 \quad (314)$$

$$H[k] = \begin{cases} 0; & x < 0 \\ 1; & x > 0 \end{cases} \quad (315)$$

The other transformation we require is given by the Mellin integral:

$$M[\alpha] = \int_0^\infty z^{\alpha-1} f(z) dz \quad (316)$$

One immediately obtains the p-adic sequences:

$$M_{FLP}[\alpha] = M[F_P(z)], \quad M_{FLQ}[\alpha] = M[F_Q(z)]$$

$$M_{FLP}[\alpha] = M[F_p(z)],$$

$$M_{FLP}[\alpha] = \frac{i\pi}{4} \Gamma(\alpha) \left\{ \frac{1}{4^\alpha} - \frac{1}{3^\alpha} + \frac{2}{2^\alpha} + 1 \right\} + \frac{i\pi \Xi(\alpha)}{4} \quad (317)$$

$$M_{FLP}[\alpha] = \frac{i\pi}{4} \Gamma(\alpha) \left\{ -\frac{1}{4^\alpha} + \frac{1}{3^\alpha} + \frac{2}{2^\alpha} + 3 \right\} - \frac{i\pi \Xi(\alpha)}{4} \quad (318)$$

$$M[F_Q(z)] = \frac{i\pi}{16} \Gamma(\alpha) \left\{ -\frac{1}{7^\alpha} + \frac{2}{6^\alpha} + \frac{1}{5^\alpha} + \frac{22}{2^\alpha} + 1 \right\} + \frac{i\pi \Xi(\alpha)}{2} \quad (319)$$

Mellin integrating the original Laplace transform, we find for $|\alpha| < 1$:

$$M_{LP}[\alpha] = \frac{\pi}{8} \frac{Z(\alpha)}{\sin\left(\frac{\pi(\alpha-1)}{2}\right)} \quad (320)$$

$$M_{LQ}[\alpha] = \frac{\pi}{96} \frac{Y(\alpha)}{\sin\left(\frac{\pi(\alpha-1)}{2}\right)} \quad (321)$$

$$M_{Lp}[\alpha] = -\frac{\pi}{8} \frac{X(\alpha)}{\sin\left(\frac{\pi(\alpha-1)}{2}\right)} \quad (322)$$

$$X(\alpha) = \left(-3 + 2^\alpha - 3^{\alpha-1} + 2^{2(\alpha-1)}\right) \quad (323)$$

$$Y(\alpha) = (3 + 33 \times 2^\alpha - 3^\alpha + 3 \times 5^{(\alpha-1)} + 6^\alpha - 3 \times 7^{(\alpha-1)}) \quad (324)$$

$$Z(\alpha) = \left(1 + 2^\alpha - 3^{\alpha-1} + 2^{2(\alpha-1)}\right) \quad (325)$$

The other rational function has a pole at the origin. In this description of the space the Fourier, Laplace, Mellin integrals and cosine transform play a complementary role to the unitary matrices used earlier. The order of integration is compositional, in that we are fully integrating over the variables at each time and the order must not be interchanged except in unique circumstances. These functions, and their related theta transforms are part of a general group of elliptic symmetry operators. One may plot these simply on a polar plot diagram.

15 Semigroup Transformations

Consider the unitary matrices $\{\hat{Q}, \hat{D}, \hat{J}\}$. We may write these matrices as the outer product of unit eigenvectors with column vectors:

$$\hat{Q} = \begin{bmatrix} \vdots & \vdots & \vdots \\ q_1 & q_2 & q_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = [|q_1\rangle \quad |q_2\rangle \quad |q_3\rangle] \quad (326)$$

$$\hat{J} = \begin{bmatrix} \vdots & \vdots & \vdots \\ j_1 & j_2 & j_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = [|j_1\rangle \quad |j_2\rangle \quad |j_3\rangle] \quad (327)$$

$$\hat{D} = \begin{bmatrix} \vdots & \vdots & \vdots \\ d_1 & d_2 & d_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = [|d_1\rangle \quad |d_2\rangle \quad |d_3\rangle] \quad (328)$$

Writing $|A_l(s)\rangle_m = [\hat{A}(s)]_{l,m} = \Psi_{l,m}^A(s)$. We first consider the various transformations on the unitaries:

$$\hat{L}^{a_i}(\alpha) |a_i(\sigma)\rangle = |a_i(\sigma + \alpha)\rangle \quad (329)$$

Finding the matrix operators:

$$\{\hat{L}^D\} = \begin{bmatrix} \cos\sigma & 0 & \sin\sigma \\ 0 & 1 & 0 \\ -\sin\sigma & 0 & \cos\sigma \end{bmatrix} \& \begin{bmatrix} \cos\sigma & 0 & -\sin\sigma \\ 0 & 1 & 0 \\ +\sin\sigma & 0 & \cos\sigma \end{bmatrix} \quad (330)$$

$$\{\hat{L}^Q\} = \begin{bmatrix} \cos\sigma & 0 & ie^{-i\theta}\sin\sigma \\ 0 & 1 & 0 \\ ie^{i\theta}\sin\sigma & 0 & \cos\sigma \end{bmatrix},$$

$$\& \begin{bmatrix} \cos\sigma & 0 & ie^{-i\theta}\sin\sigma \\ 0 & 1 & 0 \\ -ie^{i\theta}\sin\sigma & 0 & \cos\sigma \end{bmatrix} \quad (331)$$

$$\{\hat{L}^J\} = \begin{bmatrix} \cos\sigma & 0 & i\sin\sigma \\ 0 & 1 & 0 \\ i\sin\sigma & 0 & \cos\sigma \end{bmatrix} \& \begin{bmatrix} \cos\sigma & 0 & -i\sin\sigma \\ 0 & 1 & 0 \\ -i\sin\sigma & 0 & \cos\sigma \end{bmatrix}$$

$$\& \begin{bmatrix} \cos\sigma & 0 & -i\sin\sigma \\ 0 & 1 & 0 \\ i\sin\sigma & 0 & \cos\sigma \end{bmatrix} \quad (332)$$

16 Elliptic Wave Equations

Writing $|A_l(s)\rangle_m = [\hat{A}(s)]_{l,m} = \Psi_{l,m}^A(s)$, we now consider various operators on the respective wave-functions. It is important to note that the variable may properly be held to be a periodic complex variable, and the function extended meromorphically to the complex space. We will work only on the section where the wave is continuous, invertible and free of poles, so we may multiply and divide by trigonometric functions. In this section, as we are dealing with changes of variables, we must use partial differential operators to distinguish them from the strict time derivative treated elsewhere in the paper. Let us examine a particular co-ordinate transform, to show how we may write down the wave equations. We have:

$$[D(\chi)]_{3,1} = \frac{i}{\sqrt{2}}\sin(\chi) = \tau_{3,1}^D = \tau \quad (333)$$

where we drop the subscripts for convenience. Directly differentiating:

$$\frac{d\tau}{d\chi} = \frac{i}{\sqrt{2}} \cos\chi$$

Squaring the derivative:

$$(d\tau)^2 = -\left(\frac{1}{2} + \tau^2\right)d\chi^2$$

We find after inverting the fraction and taking the square root:

$$\frac{d\chi}{d\tau} = \frac{\partial\chi}{\partial\tau} = \frac{-i}{\sqrt{\frac{1}{2} + \tau^2}} \quad (334)$$

Partial differentiating a function twice with respect to our parameter:

$$\frac{\partial^2\Psi}{\partial\tau^2} = \left(\frac{\partial\chi}{\partial\tau}\right)^2 \frac{\partial^2\Psi}{\partial\chi^2} + \frac{\partial\Psi}{\partial\chi} \frac{\partial}{\partial\tau} \left(\frac{\partial\chi}{\partial\tau}\right) \quad (335)$$

Substitution of the appropriate values brings:

$$\frac{\partial^2\Psi}{\partial\tau^2} = \frac{1}{\frac{1}{2} + \tau^2} \left(-\frac{\partial^2\Psi}{\partial\chi^2} + \frac{i\tau}{\sqrt{\frac{1}{2} + \tau^2}} \frac{\partial\Psi}{\partial\chi} \right) \quad (336)$$

We can then find the differential-integral equations for the wave-function. In general it takes the form:

$$\Psi_{i,j}^k[\sigma] = \Psi_{i,j}^k[0] + \epsilon_{i,j}^k \int_0^\sigma d\nu \hat{V}(\nu) \Psi_{i,j}^k[\nu] \quad (337)$$

$$\hat{V}(\nu) = F_1^{i,j,k}(\nu) \left(\mu_{i,j,k} F_2^{i,j,k}(\nu) + \beta_{i,j}^k \nabla_\nu^2 \right) \quad (338)$$

$$\epsilon_{i,j}^k \in \{1, -1, i, -i\}; \beta_{i,j}^k \in \{1, -1\}; \nabla_u^2 = \frac{\partial^2}{\partial u^2} \quad (339)$$

$$(F_1^{i,j,k}, F_2^{i,j,k}) \in \{(\cot\nu, \cos^2\nu), (\tan\nu, \sin^2\nu)\} \quad (340)$$

Our entire wavefunction is then:

$$\Psi_{i,j}^k[s, \tau_{i,j}^k] = \Gamma_{i,j}^k[s] \Psi_{i,j}^k[\tau_{i,j}^k] \quad (341)$$

$$\frac{\partial^2 \Gamma_{i,j}^k[s]}{\partial s^2} = \mu_{i,j,k} \Gamma_{i,j}^k[s] \quad (342)$$

These formulae are generated by a group of partial differential equations of type:

$$(z^2 + a^2) \frac{\partial^2 \Psi}{\partial z^2} = - \frac{\partial^2 \Psi}{\partial \chi^2} + \frac{iz}{\sqrt{z^2 + a^2}} \frac{\partial \Psi}{\partial \chi} \quad (343)$$

$$z = f(\chi), |z| < a, f(\chi + m\omega) = f(\chi) \quad (344)$$

and we take the angular variable so as to avoid any discontinuities in the relevant trigonometric ratios other than at the extremum, where we are guaranteed analytic continuity by the hyperbolic sine-sine relationship. They are on the perimeter of the wavefunction; we are guaranteed continuity and unitarity by virtue of our matrix construction. One can label the functions by their $F_1^{i,j,k}$, $\epsilon_{i,j}^k$ and $\beta_{i,j}^k$. If we consider these functions as proper analytic functions in the complex plane, it is straightforward to derive various differential systems, analogous to the Cauchy-Riemann equations. For example, if $z = re^{i\theta}$, $f(z) = u(z) + iv(z)$, we obtain the cylindrical polar set:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad (345)$$

$$\frac{\partial v}{\partial r} = - \frac{1}{r} \frac{\partial u}{\partial \theta} \quad (346)$$

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} = 0 \quad (347)$$

If $z = r \cos \theta$, $f(z) = u(z) + iv(z)$, we find the symmetrical equations:

$$\frac{\partial u}{\partial r} = - \cot \theta \frac{\partial v}{\partial \theta} \quad (348)$$

$$\frac{\partial v}{\partial r} = - \cot \theta \frac{\partial u}{\partial \theta} \quad (349)$$

If $z = ir \cos \theta$, $f(z) = u(z) + iv(z)$, we find polar elliptic equations:

$$r \frac{\partial u}{\partial r} = - \frac{z}{\sqrt{z^2 + r^2}} \frac{\partial v}{\partial \theta} \quad (350)$$

$$r \frac{\partial v}{\partial r} = \frac{z}{\sqrt{z^2 + r^2}} \frac{\partial u}{\partial \theta} \quad (351)$$

Differentiating the correct parameterisation, analytically continuing it to the complex plane and using separation of variables allows us to solve for the ground state of the system, which can be quite difficult. The radial part of the differential equation is generally the circular Bessel function:

$$\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \frac{\alpha}{r^2} R(r) = 0 \quad (352)$$

The angular part of the particular solution is then:

$$\Psi(r, \theta) = R(r)Y(\theta) \quad (353)$$

$$\frac{d^2 Y(\theta)}{d\theta^2} + F_1(\theta) \frac{dY(\theta)}{d\theta} + \alpha F_2(\theta) Y(\theta) = 0 \quad (354)$$

17 Integral Relations

Expanding the Hamiltonian matrix in the spectral representation, we find:

$$\tilde{H} = \sum_n E_n |\phi_n\rangle \langle \phi_n| \quad (355)$$

In this particular situation, the eigenenergies are equally separated from zero, we may therefore write:

$$\tilde{H} = E(|\phi_+\rangle \langle \phi_+| - |\phi_-\rangle \langle \phi_-|) \quad (356)$$

An alternative way of writing this Hamiltonian is:

$$\tilde{H} = \sum_{i,j} \epsilon_{ij}(t) \hat{Q}_{ij} + \epsilon_{ij}^*(t) \hat{Q}_{ij}^T \quad (357)$$

$$\hat{Q}_{ij} = |i\rangle \langle j| \quad (358)$$

Using the dynamical equations of motion,

$$i \frac{d}{dt} \begin{bmatrix} |\phi_+\rangle \\ |\phi_-\rangle \end{bmatrix} = E \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} |\phi_+\rangle \\ |\phi_-\rangle \end{bmatrix} \quad (359)$$

Let us now expand the wave-vector in the eigenbasis:

$$i \frac{d}{dt} \left(\sum_j c_j(t) |j(t)\rangle \right) = \sum_j E_j |j(t)\rangle \quad (360)$$

Dropping the sums for convenience and assuming Einstein summation convention, we evaluate the differential equation:

$$i c_j(t) \frac{d}{dt} |j(t)\rangle = (E_j - i \frac{dc_j}{dt}) |j(t)\rangle \quad (361)$$

Consider a path integral on our state space:

$$[K_{jk}(T, 0)] = \int_{\tilde{x}_j(0)}^{\tilde{x}_k(T)} \exp(-i \int_0^T ds \hat{\mathcal{L}}[E_j, c_j|s]) D[\tilde{x}_\alpha(s)] \quad (362)$$

We may write the Lagrangian density as the formula:

$$\hat{\mathcal{L}} = -E_j \dot{c}_j(s) \delta_{jk} + i c_j(s) \langle k(s) | \tilde{H}(s) | j(s) \rangle \quad (363)$$

Up to isomorphism, this fulfils an equivalent role to the time evolution operator. As we have managed to already find a number of differential relationships, the necessary question is to enquire as to whether there are a series of equivalent

integral identities that we may use to evaluate these particular groups of elliptic differential equations. Consider a simple toy problem of a wave-packet interacting with a time dependent oscillating potential:

$$i\hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + V_0 \cos \omega t \cdot \Psi(x, t) \quad (364)$$

Using natural units and rescaling the time, we may write this as a neat operator equation:

$$i \frac{\partial \Psi(x, t)}{\partial t} = (\hat{H}_0 + \hat{H}_I(t)) \Psi(x, t) \quad (365)$$

Now, if we cosine-transform the time co-ordinate, our equations of motion transform:

$$z = \cos \omega t \quad (366)$$

$$t = \frac{1}{\omega} \cos^{-1}(z) \quad (367)$$

$$\hat{H}_I(t) \mapsto \hat{H}_I(z) = V_0 z \quad (368)$$

$$\hat{H}_0 \mapsto \hat{H}_0 \quad (369)$$

$$\frac{\partial}{\partial t} \mapsto \frac{\partial z}{\partial t} \frac{\partial}{\partial z} = \omega \sqrt{1-z^2} \frac{\partial}{\partial z} \quad (370)$$

Writing our original wave-equation in the transformed frame, we find:

$$i\omega \sqrt{1-z^2} \frac{\partial \Psi(x, z)}{\partial z} = -\frac{1}{2} \frac{\partial^2 \Psi(x, z)}{\partial x^2} + V_0 z \Psi(x, z) \quad (371)$$

Using the method of separation of variables, we write an ansatz for our wave-function:

$$\Psi(x, z) = \phi(x) \beta(z) \quad (372)$$

Finding the equivalent set of equations:

$$E \phi(x) = -\frac{1}{2} \frac{d^2 \phi(x)}{dx^2} \quad (373)$$

$$i\omega \frac{d\beta(z)}{dz} = \frac{(E - V_0 z)}{\sqrt{1-z^2}} \beta(z) \quad (374)$$

The first of these equations is readily solvable using complex exponentials, however, the second presents us with some difficulties. Firstly, we rescale the variables to place the differential equation into dimensionless form, finding:

$$i \frac{d\beta(z)}{dz} = \frac{(1 - \alpha z)}{\sqrt{1-z^2}} \beta(z) \quad (375)$$

Directly differentiating:

$$(1-z^2) \frac{d^2 \beta}{dz^2} = (p(\alpha, z) - \frac{iq(\alpha, z)}{\sqrt{1-z^2}}) \beta \quad (376)$$

where the p's and q's are polynomial functions. This is one of our basic elliptic differential equations. Note the useful integral formula:

$$\int_a^b \sec z(1 + \tan z)dz = \int_{y_a}^{y_b} \frac{dy}{\sqrt{y^2 - 1}} + i(y_b - y_a) \quad (377)$$

where $y = \sec z$. To consider further transformations, consider first one of the polynomials from the previous section:

$$b_1(z) = z^8 + 29z^6 + 208z^4 + 306z^2 - 144 \quad (378)$$

It is possible to completely specify this polynomial by its root set, viz.:

$$b_1(z) = 0, z \in \{\pm\alpha_1, \pm i\beta_1, \pm i\delta_1, \pm i\gamma_1\} \quad (379)$$

Moving sequentially through the other relevant polynomials, we find:

$$b_2(z) = z^7 - 125z^5 + 25z^4 + 192z^3 - 960z^2 - 2880z + 20160 \quad (380)$$

$$b_2(z) = 0, z \in \{\alpha_2, \beta_2, -\gamma_2, z_1, z_1^*, z_2, z_2^*\} \quad (381)$$

This point set has the neat property of being a pentagram in the complex plane with three non-symmetrical points on the real axis, one of which is the primary vertice of the pentagon.

$$b_3(z) = z^6 + 120z^5 + 5016z^4 + 86527z^3 + 550413z^2 + 896923z + 396900 \quad (382)$$

This polynomial has its root set confined to the negative real axis:

$$b_3(z) = 0, z \in \{-\alpha_3, -\beta_3, -\gamma_3, -\delta_3, -\epsilon_3, -\zeta_3\} \quad (383)$$

Finally, consider the polynomial $b_4(z) = b_1(\sqrt{z})$. As the original polynomial is even in z , the new square-root transformed polynomial is of the form:

$$b_4(z) = z^4 + 29z^3 + 208z^2 + 306z - 144 \quad (384)$$

which has a root set:

$$b_4(z) = 0, z \in \{\alpha_4, -\beta_4, -\gamma_4, -\delta_4\} \quad (385)$$

By examining the cosine and exponentially transformed polynomials, $b_j(\cos \theta)$ & $b_j(\exp i\theta)$, and real and imaginary parts thereof, it is a simple exercise to develop a series of polar representations for these polynomial functions. The complex analytic behaviour is generated by these or similar sets of roulettes. They are related to a number of interesting integral relations. We begin with a simple example:

$$\int_0^{2\pi} \frac{b_4(u)du}{\sqrt{1-u^2}} = ia + Q(\pi)\sqrt{4\pi^2 - 1} \quad (386)$$

The other polynomials have similar identities with respect to this measure:

$$\int_{-1}^{+1} \frac{b_1(u)du}{\sqrt{1-u^2}} = \frac{12331\pi}{128} \quad (387)$$

$$\int_0^{3\pi} b_1(u)\sqrt{1-u^2}du = b\pi + i\sqrt{9\pi^2-1}(c + (9\pi^2-1)P(\pi)) \quad (388)$$

$$P(\pi) = \sum_{n=0}^3 C_n \pi^{2n+1} \quad (389)$$

$$\int_0^{2\pi} b_1(u)\sqrt{1-u^2}du = k\pi + i\sqrt{4\pi^2-1}(l + (4\pi^2-1)Y(\pi)) \quad (390)$$

$$Y(\pi) = \sum_{n=0}^3 D_n \pi^{2n+1} \quad (391)$$

$$\{D_n\} \subseteq \{C_n\}, D_n = 0 \pmod{C_j} \quad (392)$$

$$\int_0^{\pi/2} b_1(u)\sqrt{1-u^2}du = g\pi + i\sqrt{\pi^2-4}(f + (\pi^2-4)r(\pi)) \quad (393)$$

$$r(\pi) = \sum_{n=0}^3 M_n \pi^{2n+1} \quad (394)$$

$$\{M_n\} \subseteq \{C_n\} \quad (395)$$

$$\int_{-l}^{+l} b_1(u)\sqrt{1-u^2}du = \sqrt{1-l^2}\left(\sum_0^3 C_n l^{2n+1}\right) - m \sin^{-1}(l) \quad (396)$$

$$\int_0^a \frac{b_1(u)du}{\sqrt{1-u^2}} = ip(a)\sqrt{a^2-1} + \lambda a + i \sin^{-1}(a) \quad (397)$$

Off the unit circle, these integrals lose analyticity unless we modify the weighting function. One way to do this is by defining a co-distribution:

$$\int_{-1}^{+1} du(1-u^2)^{\alpha/2} = \sqrt{\pi} \frac{\Gamma(\frac{\alpha}{2}+1)}{\Gamma(\frac{\alpha}{2}+\frac{3}{2})} \quad (398)$$

We then form marginal distributions:

$$p_{k,W}(\alpha) = \frac{\int_{\Omega} p_k(s)W(s,\alpha)ds}{\int_{\Omega} W(s,\alpha)ds} \quad (399)$$

$$b_{1,W}(\alpha) = -3 \frac{\sum q_n \alpha^n}{\prod(\alpha+2(n+1)+1)} = -3 \frac{p(\alpha)}{q(\alpha)} \quad (400)$$

Using partial fractions to expand the denominator:

$$\frac{1}{\prod(\alpha + 2(n+1) + 1)} = \frac{1}{48} \left(\frac{1}{\alpha + 3} - \frac{1}{\alpha + 9} \right) + \frac{1}{16} \left(\frac{1}{\alpha + 7} - \frac{1}{\alpha + 5} \right) \quad (401)$$

$$p(\alpha) = 48\alpha^4 + 1050\alpha^3 + 7538\alpha^2 + 17653\alpha - 1214 \quad (402)$$

$$\lim_{\alpha \rightarrow \infty} b_{1,W}(\alpha) = -144 \quad (403)$$

This is not the only co-distribution, we may form an equivalent functional:

$$W'(\alpha, s) = (1 - \alpha^2)^{-s/2} = W(-s, \alpha) \quad (404)$$

These integral relationships are then neatly expressible in terms of hypergeometric functions:

$$\int d\alpha W(-s, \alpha) = \alpha \left\{ \begin{matrix} \frac{1}{2} & \alpha^2 \\ \frac{s}{2} & \frac{3}{2} \end{matrix} \right\} \quad (405)$$

$$\int d\alpha W(-s, \alpha) b_{1,W}(\alpha) = \sum_{n=1}^5 \left\{ \begin{matrix} 1/2 & \alpha^2 \\ \frac{s}{2} & \frac{n}{2} + 1 \end{matrix} \right\} \alpha^n C_n \quad (406)$$

The point symmetry group of $p(\alpha)$ is a kite, all the zeroes of $q(\alpha)$ lie along the negative real axis.

18 Chebyshev Polynomials

Consider a unit triangle in the complex plane. Using De-Moivre's theorem:

$$(e^{i\theta})^m = (\cos \theta + i \sin \theta)^m = \cos m\theta + i \sin m\theta \quad (407)$$

We may write an alternative transformation, viz.:

$$z = x + iy = x + i\sqrt{1-x^2} \quad (408)$$

The left-hand side of De-Moivre's theorem then reads as:

$$(x + i\sqrt{1-x^2})^m = \cos m\theta + i \sin m\theta \quad (409)$$

We define the Chebyshev polynomial of the first kind:

$$T_m(\cos \theta) = \cos m\theta \quad (410)$$

Using the double angle formula, it is a simple exercise to derive the first recursion relation:

$$T_m(x) = xT_{m-1}(x) + (x^2 - 1)U_{m-1}(x) \quad (411)$$

where we have defined the Chebyshev polynomial of the second kind:

$$U_m(\cos \theta) = \frac{\sin m\theta}{\sin \theta} \quad (412)$$

with recursion relation:

$$U_m(x) = xU_{m-1}(x) + T_{m-1} \quad (413)$$

Directly differentiating the recursion relations, we obtain the system:

$$\frac{dT_m(x)}{dx} = mU_m(x) \quad (414)$$

$$(1-x^2)\frac{dU_m}{dx} = m(xU_m - T_m) \quad (415)$$

$$(1-x^2)\frac{d^2T_m}{dx^2} = (mxU_m - m^2T_m) \quad (416)$$

$$(1-x^2)\frac{d^2T_m}{dx^2} - x\frac{dT_m}{dx} + m^2T_m = 0 \quad (417)$$

This differential equation is related to the elliptic wave equations used in the previous sections. This comes about due to our particular break-down of the wave-function in terms of roots of unity on the unit circle.

19 Circular Co-ordinates

Consider the standard cylindrical co-ordinate system in two dimensions, i.e. a circular system:

$$\vec{x} = r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (418)$$

Directly differentiating to find the velocity vector:

$$\frac{d\vec{x}}{dt} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \quad (419)$$

The polar unit vectors have representation:

$$\hat{e}_r(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \hat{e}_\theta(\theta) = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \quad (420)$$

$$\hat{e}_r(\theta) \cdot \hat{e}_\theta(\theta) = 0 \quad (421)$$

$$\hat{e}_{r,\theta}(\theta + \alpha) = \hat{L}(\alpha)\hat{e}_{r,\theta}(\theta) \quad (422)$$

$$\hat{L}(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \quad (423)$$

Writing out the arc-length for the system:

$$ds^2 = dr^2 + r^2d\theta^2 \quad (424)$$

we derive the gradient for the wave-function:

$$\nabla\Psi = \hat{e}_r \frac{\partial\Psi}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial\Psi}{\partial\theta} \quad (425)$$

Deriving the Laplacian in circular co-ordinates:

$$\nabla^2\Psi = (\nabla \cdot \nabla)\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r} \frac{\partial\Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2\Psi}{\partial\theta^2} \quad (426)$$

The time-independent wave equation for a free particle on the circle is given by

$$\hat{H}\Psi(r, \theta) = -\nabla^2\Psi(r, \theta) = \xi\Psi(r, \theta) \quad (427)$$

Using the method of separation of variables, we obtain the equivalent system:

$$\Psi(r, \theta) = R(r)Y(\theta) \quad (428)$$

$$\frac{d^2Y(\theta)}{d\theta^2} = -k^2Y(\theta) \quad (429)$$

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \left(\frac{k^2}{r^2} - \xi\right)R(r) = 0 \quad (430)$$

These equations have the well-known solutions of complex exponentials and Bessel functions. The angular part obeys periodic boundary conditions, and can be separated into an odd and even part, being the sine and cosine:

$$Y(\theta) = Ae^{ik\theta} + Be^{-ik\theta} \quad (431)$$

$$Y_\pm(\theta) = \pm Y_\pm(-\theta) \quad (432)$$

$$Y(2\pi) = Y(0) \quad (433)$$

finding a particular quantisation for the system as given by:

$$k = \pm n \quad (434)$$

Rewriting the radial wave equation in natural units $\xi = 1$, we find:

$$r^2 \frac{d^2R_n}{dr^2} + r \frac{dR_n}{dr} + (n^2 - r^2)R_n = 0 \quad (435)$$

Our solution is then the Bessel function:

$$R_n(r) = J_n(r) \quad (436)$$

This has an integral expansion and inner product:

$$J_n(r) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-i(n\phi - r \sin \phi)} d\phi \quad (437)$$

$$\int_{-\pi}^{+\pi} J_m^*(\nu) J_n(\nu) d\nu = \frac{\sin(\pi(m-n))}{2\pi^2(m-n)} = \frac{\delta(m-n)\delta_{mn}}{2\pi^2} \quad (438)$$

20 Time Dependent Oscillator Revisited

Take the wave equation in one space dimension, coupled to a time dependent oscillator:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m_e} \frac{\partial^2 \Psi}{\partial x^2} + V_0 \cos \omega t \Psi \quad (439)$$

Using the cosine transform, and the method of separation of variables:

$$\Psi(x, t) = \phi(x)\beta(t) \quad (440)$$

$$z = \cos \omega t \Leftrightarrow t = \frac{1}{\omega} \cos^{-1}(z) \quad (441)$$

we find the differential equation of state:

$$i \frac{d\beta}{\beta} = \frac{(1 - \alpha z)}{\sqrt{1 - z^2}} dz \quad (442)$$

which has an explicit solution:

$$i(\ln \beta(z) - \ln \beta_0) = -\cos^{-1}(z) + \alpha \sqrt{1 - z^2} \quad (443)$$

This is mid-way between a Bessel function and a Chebyshev polynomial. Writing $\chi = \omega t$, $z = \cos \chi$ we may show a neat set of transformations:

$$\frac{\partial \Psi}{\partial \chi} = -\sqrt{1 - z^2} \frac{\partial \Psi}{\partial z} \quad (444)$$

$$\frac{\partial \Psi}{\partial z} = \frac{-1}{\sqrt{1 - z^2}} \frac{\partial \Psi}{\partial \chi} = \frac{-1}{\sin \chi} \frac{\partial \Psi}{\partial \chi} \quad (445)$$

Differentiating these expressions directly, we arrive at two equivalent differential equations:

$$\frac{\partial^2 \Psi}{\partial \chi^2} = \sqrt{1 - z^2} \frac{\partial}{\partial z} (\sqrt{1 - z^2} \Psi) \quad (446)$$

$$\frac{\partial^2 \Psi}{\partial z^2} = \frac{1}{\sin \chi} \frac{\partial}{\partial \chi} \left(\frac{1}{\sin \chi} \frac{\partial \Psi}{\partial \chi} \right) \quad (447)$$

Expanding these, we find the pair of second order differential equations, related by the transform pair:

$$\frac{\partial^2 \Psi}{\partial \chi^2} = (1 - z^2) \frac{\partial^2 \Psi}{\partial z^2} - z \frac{\partial \Psi}{\partial z} \quad (448)$$

$$(1 - z^2) \frac{\partial^2 \Psi}{\partial z^2} = \frac{\partial^2 \Psi}{\partial \chi^2} - \cot \chi \frac{\partial \Psi}{\partial \chi} \quad (449)$$

$$z \frac{\partial \Psi}{\partial z} = \cot \chi \frac{\partial \Psi}{\partial \chi} \quad (450)$$

Taking the equation of state and directly differentiating we find a second order differential equation:

$$i(1-z^2)\frac{d^2\beta}{dz^2} = \beta(-i(1-\alpha z)^2 - \alpha\sqrt{1-z^2} + \frac{z(1-\alpha z)}{\sqrt{1-z^2}}) \quad (451)$$

This may be rewritten in the equivalent form:

$$(1-z^2)\frac{d^2\beta}{dz^2} + \left(\frac{\alpha(1-z^2)}{1-\alpha z} - z\right)\frac{d\beta}{dz} + (1-\alpha z)^2\beta = 0 \quad (452)$$

$$\beta(z) = \beta_0 \exp(i \cos^{-1}(z)) \exp(-i\alpha\sqrt{1-z^2}) \quad (453)$$

21 Discrete Fourier Transform For SU(3)

The discrete Fourier transform (DFT) on a qutrit has subtleties that are not apparent in the even dimensional case, as there are several transformations which have similar properties to the expected form. As the structure of the entire unitary space can be broken down into three sets of orthogonal rotations and two diagonal operators, defining exactly what is meant by 'odd' and 'even' is not quite so simple. This is due to the underlying group structure, as we may define a third set of properties, namely neutral. We would expect to have three subgroups which map in a similar way to the DFT on even dimensions, and one total unitary; this is indeed the case. To show that this is true, we first generate a cube root of unity:

$$z = \frac{1}{2}(-1 + i\sqrt{3}) \quad (454)$$

with complex conjugate:

$$z^* = \frac{1}{2}(-1 - i\sqrt{3}) \quad (455)$$

This has unit modulus and is a group under $\{*,^2\}$:

$$|z|^2 = 1 \quad (456)$$

$$z^2 = z^*, z = (z^*)^2 \quad (457)$$

Defining our first DFT matrix:

$$\hat{R} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & z & z^* \\ 1 & z^* & z \end{bmatrix} \cong \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & z & z^2 \\ 1 & z^2 & z^4 \end{bmatrix} \quad (458)$$

This matrix is unitary:

$$\hat{R}\hat{R}^\dagger = \hat{R}^\dagger\hat{R} = \hat{\mathbf{1}} \quad (459)$$

and is a fourth-root of unity $\hat{R}^4 = \hat{1}$. One useful relationship is:

$$\hat{R}^T \hat{R} = \hat{R} \hat{R}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (460)$$

which is the equivalent C-NOT for qutrits. We may write our complex unit as the cube root of unity:

$$z^3 = 1 \quad (461)$$

with equivalent Euler representation:

$$z = \exp\left(\frac{2\pi i}{3}\right) \quad (462)$$

Let us now consider a particular decomposition of the Hamiltonian:

$$\hat{H} = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix} + \hat{Q} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{Q}^\dagger = \tilde{H}_0 + \tilde{H}_Q \quad (463)$$

We define an odd-valued split DFT via:

$$\hat{Q} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & k & k^3 \\ 1 & k^3 & k^5 \end{bmatrix}, \hat{Q}^\dagger = \begin{bmatrix} 1 & 1 & 1 \\ 1 & k^{-1} & k^{-3} \\ 1 & k^{-3} & k^{-5} \end{bmatrix} \quad (464)$$

with function values:

$$k = e^{i\theta}, k^{-1} = k^* \quad (465)$$

This takes our Hamiltonian to the operator:

$$\hat{H} = \begin{bmatrix} \omega_1 & 1 - k^{-1} & 1 - k^{-3} \\ 1 - k & \omega_2 & 1 - k^{-2} \\ 1 - k^3 & 1 - k^2 & \omega_3 \end{bmatrix} \quad (466)$$

Writing out the formulae for the trigonometric polynomials:

$$k^3 = \cos \theta (4 \cos^2 \theta - 3) + i \sin \theta (4 \sin^2 \theta - 3) \quad (467)$$

$$k^2 = 2 \cos^2 \theta - 1 + i 2 \sin \theta \cos \theta \quad (468)$$

$$k^{-n} = (k^n)^* \quad (469)$$

Note that this takes the Hamiltonian to the off-diagonal non-comutative submanifold; consequently we expect to find two equivalent different split DFTs which take the Hamiltonian to the same type of submanifolds. This type of transformation will not be unitary or invertible, it is more along the same lines

as making a projection onto a co-ordinate axis, keeping a subset of the co-ordinates and discarding the rest of the set. Writing out the transformation law on the Hamiltonian matrix:

$$\hat{H}_W = \hat{W} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{W}^\dagger \quad (470)$$

We now substitute the even-valued split DFT into this relation:

$$\hat{W} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w^4 \end{bmatrix} \quad (471)$$

and find the explicitly transformed matrix:

$$\hat{H}_W = \begin{bmatrix} 0 & 1 - w^{-1} & 1 - w^{-2} \\ 1 - w & 0 & 1 - w^{-1} \\ 1 - w^2 & 1 - w & 0 \end{bmatrix} \quad (472)$$

$$w = e^{i\chi} \quad (473)$$

This matrix has two independent component functions, thereby satisfying the physical conditions of the $SU(3)/SU(2)$ submanifold. We may write this in the form:

$$\tilde{H}_W = \hat{A} + \hat{A}^T + \hat{X} \cos \chi + \hat{Y} \sin \chi + \hat{K} + \hat{K}^\dagger \quad (474)$$

Consider the operators \hat{X}, \hat{Y} :

$$\hat{X} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \hat{Y} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad (475)$$

They form a non-commutative subgroup that is similar to $SU(2)$:

$$[\hat{X}, \hat{Y}] = 2i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = 2i\hat{Z} \quad (476)$$

$$[\hat{X}, \hat{Z}] = -i\hat{Y} \quad (477)$$

$$[\hat{Y}, \hat{Z}] = i\hat{X} \quad (478)$$

Up to a constant matrix, this Hamiltonian is equivalent to the $SO(3)$ subalgebra of $SU(3)$. The constant matrix is the set of upper and lower projectors, with representation:

$$\hat{A} = \hat{A}_1 + \hat{A}_2 + \hat{A}_3 \quad (479)$$

$$\hat{A}_1 = |1\rangle\langle 2|, \hat{A}_2 = |1\rangle\langle 3|, \hat{A}_3 = |2\rangle\langle 3| \quad (480)$$

$$\hat{A}_j^\dagger = \hat{A}_j^T \quad (481)$$

This matrix is not unique, one may define another matrix, given by:

$$\hat{J} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j^3 \end{bmatrix} \quad (482)$$

Transforming the Hamiltonian we find:

$$\hat{H}_J = \hat{J} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{J}^\dagger \quad (483)$$

$$\hat{H}_J = \begin{bmatrix} 0 & 1-j^{-1} & 1-j^{-2} \\ 1-j & 0 & 1-j^{-1} \\ 1-j^2 & 1-j & 0 \end{bmatrix} \quad (484)$$

which means our mapped Hamiltonian is an equivalent two-parameter subgroup. Writing this as a congruence relation:

$$\hat{H}_Q \not\cong \hat{H}_J \cong \hat{H}_W \quad (485)$$

The functions involved have decompositions that may be readily calculated with De-Moivre's theorem:

$$1 - j^2 = 1 - e^{2i\theta} = 1 - \cos 2\theta - i \sin 2\theta \quad (486)$$

$$1 - \cos 2\theta - i \sin 2\theta = 2(\cos^2 \theta - 1) - i2 \sin \theta \cos \theta \quad (487)$$

$$1 - j^2 = 2(\sin^2 \theta) - i2 \sin \theta \cos \theta \quad (488)$$

These functions have the necessary symmetry under conjugation to give us a Hermitian matrix:

$$j^{-n} = (j^n)^* \quad (489)$$

$$1 - j^{-l} = (1 - j^l)^* \quad (490)$$

None of the split-transforms \hat{J} , \hat{Q} or \hat{W} are unitary or orthogonal; the only matrix which has the required property to be a DFT in its own right is \hat{R} .

22 Dihedral Group On The Unit Triangle

The permutations one may apply to a triangle with uniquely labelled vertices generate a dihedral group, defined by the matrices:

$$\hat{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \hat{S}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (491)$$

$$\hat{S}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \hat{S}_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (492)$$

$$\hat{S}_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \hat{S}_6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad (493)$$

This is the smallest non-commutative group. Note that these are a group of rotofections, as opposed to proper reflections or rotation matrices. Using our Hamiltonian operators, we may define a closely related series of operators using our initial value equations:

$$\hat{Q}(0) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (494)$$

This is one of our equivalent Hadamard gates, which constitutes a pure rotation in 3-D space.

$$\hat{Q}^2(0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (495)$$

We then have a reflection in a plane of symmetry, which is a signed permutation matrix.

$$\hat{D}(0) = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix} \quad (496)$$

Because we are working on higher dimensions, there is a greater number of available operators, which we need in order to close the algebra and conduct meaningful calculations. With qubits all that is required is a phase gate, a NOT gate, a Hadamard matrix and a XOR gate.

$$\hat{D}^2(0) = \frac{1}{2} \begin{bmatrix} i-1 & i+1 & 0 \\ -i+1 & i+1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad (497)$$

$$\hat{J}(0) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix} \quad (498)$$

$$\hat{J}^4(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (499)$$

The particular matrices we have been considering above play the role of the phase gate, one of the possible equivalent C-NOT trit gates and the Hadamard gate in time optimal qutrit quantum computation. They are signed complex permutation matrices. The extremals of the unitary generators are at the points $\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ which define a base set of qutrit computational gates:

$$\hat{Q}(0) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \hat{Q}(\pi) = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (500)$$

$$\hat{Q}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 0 & ie^{-i\varrho} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ ie^{i\varrho} & -ie^{i\varrho} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad (501)$$

$$\hat{Q}\left(\frac{3\pi}{2}\right) = \begin{bmatrix} 0 & 0 & -ie^{-i\varrho} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ ie^{i\varrho} & ie^{i\varrho} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad (502)$$

It is possible to classify these matrices by their eigenvalues and determinants into various point symmetry groups.

In some sense the unitary matrices we are using play the role of “square roots” of unitary matrices themselves, which are in fact the dihedral group above. Being the complete set of positive and negative roots of the group expression, we receive twelve unique operators which may be readily used to generate a wide variety of useful operations.

$$\hat{J}(0) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix}, \hat{J}(\pi) = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -i \end{bmatrix} \quad (503)$$

$$\hat{J}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ i & -i & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \hat{J}\left(\frac{3\pi}{2}\right) = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -i & i & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad (504)$$

$$\hat{D}(0) = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix}, \hat{D}(\pi) = \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -i \end{bmatrix} \quad (505)$$

$$\hat{D}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 0 & i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \hat{D}\left(\frac{3\pi}{2}\right) = \begin{bmatrix} 0 & 0 & -i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad (506)$$

To generate these higher dimensional spin analogues, it is possible with a double handed co-ordinate system to rotate 180 degrees with the left hand, 360 degrees with the right hand and rotate the body a full 360 degrees in the same direction to return to our original position. The other fundamental isotopic spin symmetries are generated by similar arguments. The set of unitary transformations is sufficient to implement any quantum trinary computation. Some elementary operations include:

$$\hat{D}(0)\hat{D}\left(\frac{\pi}{2}\right) - \hat{D}\left(\frac{\pi}{2}\right)\hat{D}(0) = \begin{bmatrix} i & i & 0 \\ i & -i & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (507)$$

$$\hat{D}(0)\hat{D}\left(\frac{\pi}{2}\right) + \hat{D}\left(\frac{\pi}{2}\right)\hat{D}(0) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (508)$$

Other operators will be explored in future works. By classification of the eigenvalues, these matrices may be defined as a certain types of point and continuous symmetry groups in the complex plane, which are related to the roots of polynomial equations which we explored earlier. This method of qutrit computation and modelling of quantum states and control is efficient, in that it is time optimal; resource bounded, in that there is an existing physical overhead on the total energy imparted to the system, reversible, in that it is unitary, and robust to error, as it is periodic. We could consider our SU(3) system to be in all ways equivalent to the physical sum of a qubit and an ancilla. In this sense, we are moving beyond qubit quantum computation into new realms of possibility. With a qubit and an ancilla it is possible to achieve much more than with a single qubit, or any number of coupled qubits. Data correction requires ancillas, and the nature of quantum states is that all indistinguishable alternatives are to be summed over; this necessitates our examination of qutrits both as quantum computational objects in themselves, and as possible sources of error within a qubit calculation. This is the simplest example of a odd dimensional quantum state; in this particularly simple case we are fortunately able to derive rich symmetries which are descriptive of many of the full properties of groups which do not emerge in lower dimensional cases.

23 Subgroups of SU(3)

Explicitly writing out the diagonal generators of SU(3), we find:

$$\frac{1}{\sqrt{3}} \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & -2\gamma \end{bmatrix}, \begin{bmatrix} \vartheta & 0 & 0 \\ 0 & -\vartheta & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix} \quad (509)$$

The diagonal subgroup has one component which commutes with every other member of the group, the other only commutes with the rotation subgroup. We may therefore use the diagonal component to define the group structure, as it does not matter whether the commuting operator is taken as part of the constraint or Hamiltonian. The matrix structure of the possible subdivisions is then:

$$\begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix}, \begin{bmatrix} 0 & \epsilon_1 & \epsilon_2 \\ \epsilon_1^* & 0 & \epsilon_3 \\ \epsilon_2^* & \epsilon_3^* & 0 \end{bmatrix} \quad (510)$$

$$\begin{bmatrix} \omega_1 & 0 & \kappa \\ 0 & \omega_2 & 0 \\ \kappa^* & 0 & \omega_3 \end{bmatrix}, \begin{bmatrix} 0 & \epsilon_1 & 0 \\ \epsilon_1^* & 0 & \epsilon_2 \\ 0 & \epsilon_2^* & 0 \end{bmatrix} \quad (511)$$

$$\begin{bmatrix} 0 & 0 & \kappa \\ 0 & 0 & 0 \\ \kappa^* & 0 & 0 \end{bmatrix}, \begin{bmatrix} \omega_1 & \epsilon_1 & 0 \\ \epsilon_1^* & \omega_2 & \epsilon_2 \\ 0 & \epsilon_2^* & \omega_3 \end{bmatrix} \quad (512)$$

$$\begin{bmatrix} \vartheta & \beta & 0 \\ \beta^* & -\vartheta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \gamma & 0 & \epsilon_1 \\ 0 & \gamma & \epsilon_2 \\ \epsilon_1^* & \epsilon_2^* & -2\gamma \end{bmatrix}, \quad (513)$$

For any choice of Hamiltonian operator and associated constraint, we either find one or the other to be periodic (or constant). The equations of dynamic state are equivalent up to isometry with a permutation matrix.

24 Geodesic On SU(4)

Let us examine a system where we can only apply global control pulses, without individual addressing, on two qubits. The Hamiltonian matrix is the Heisenberg model:

$$\tilde{H}[t] = \lambda_x \hat{\sigma}_x \otimes \hat{\sigma}_x + \lambda_y \hat{\sigma}_y \otimes \hat{\sigma}_y + \lambda_z \hat{\sigma}_z \otimes \hat{\sigma}_z \quad (514)$$

In matrix form:

$$\tilde{H}[t] = \begin{bmatrix} \lambda_z & 0 & 0 & \lambda_- \\ 0 & -\lambda_z & \lambda_+ & 0 \\ 0 & \lambda_+ & -\lambda_z & 0 \\ \lambda_- & 0 & 0 & \lambda_z \end{bmatrix}; \lambda_{\pm} = \lambda_x \pm \lambda_y \quad (515)$$

The constraint may be expanded in the Hermitian basis:

$$\tilde{F}[t] = (\vec{m}(t) \cdot \vec{\sigma}) \otimes \hat{\mathbf{1}} + \hat{\mathbf{1}} \otimes (\vec{n}(t) \cdot \vec{\sigma}) + \sum_{i \neq j} \Xi_{i,j}(t) \hat{\sigma}_i \otimes \hat{\sigma}_j \quad (516)$$

Evaluating the quantum brachistochrone, we find the vector relationship:

$$\frac{d}{dt} \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix} = 0 \quad (517)$$

Our time optimal Hamiltonian is therefore a constant matrix, and may be exponentiated directly. Our wave-vector after some time is given by:

$$|\psi(t)\rangle = \exp(-i\tilde{H}t) |\psi(0)\rangle \quad (518)$$

Evaluating the quantum brachistochrone equation, and choosing the initial state $|\psi(0)\rangle = [1, 0, 0, 0]^T$ we immediately find $\lambda_x = -\lambda_y$. Our final state is:

$$|\psi(T)\rangle = \frac{1}{\sqrt{2}}(|00\rangle - i|11\rangle) \quad (519)$$

from which we obtain $\lambda_z = 0$. Our state evolves according to:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos(2\lambda_x t) \\ 0 \\ 0 \\ -i\sin(2\lambda_x t) \end{bmatrix} \quad (520)$$

and therefore our energy-time relationship is $T = \frac{\pi}{\lambda_x}$.

25 Dimensional Arrays Of Arbitrary Size

Take an arbitrary tracefree matrix in $SU(n)$, which we can express as the array:

$$\tilde{X} = \begin{bmatrix} \omega_1 & \alpha_1 & \alpha_2 & \cdots & \vdots \\ \alpha_1^* & \omega_2 & \alpha_3 & \cdots & \vdots \\ \alpha_2^* & \alpha_3^* & \ddots & \cdots & \alpha_{n-1} \\ \vdots & \cdots & \ddots & \ddots & \alpha_n \\ \vdots & \cdots & \alpha_{n-1}^* & \alpha_n^* & \omega_n \end{bmatrix}, \sum_j \omega_j = 0 \quad (521)$$

This can be mapped invertibly and one-to-one to the vector

$$\vec{X} = [(\omega_1, \cdots, \omega_n), (\alpha_1, \cdots, \alpha_k), (\alpha_1^*, \cdots, \alpha_k^*)]^T \quad (522)$$

which exists in the space $[\mathbb{R}^{n-1}, \mathbb{C}^k, \mathbb{C}^{k*}]^T$, and the dimension of the complex space is $k = \sum_1^{n-1} j$. One of the components of the real vector within the multivector is dependent; we retain the dummy variable to aid in the analysis. In any real calculation we immediately remove the redundant variable once the differential equations are calculated. If we choose the particular Hamiltonian matrix:

$$\tilde{H} = \begin{bmatrix} \omega_1 & 0 & \cdots & \cdots & 0 & \varepsilon_k \\ 0 & \omega_2 & \cdots & \cdots & \varepsilon_{k-1} & 0 \\ \vdots & \vdots & \cdot & \cdot & \vdots & \vdots \\ \vdots & \vdots & \cdot & \cdot & \vdots & \vdots \\ 0 & \varepsilon_{k-1}^* & \cdots & \cdots & \omega_{n-1} & 0 \\ \varepsilon_k^* & 0 & \cdots & \cdots & 0 & \omega_n \end{bmatrix} \quad (523)$$

$$\tilde{H} = \sum_n \omega_n(t) |n\rangle \langle n| + \sum_{k+j=n} (\varepsilon_j(t) |k\rangle \langle j| + \text{h.c.}) \quad (524)$$

Then the constraint is given by:

$$\tilde{F} = \sum_{i \neq j, i+j \neq n} (\alpha_{i,j}(t) |i\rangle \langle j| + \text{h.c.}) \quad (525)$$

$$\tilde{F} = \begin{bmatrix} 0 & \beta_{1,2} & \cdots & \cdots & \beta_{1,n-1} & 0 \\ \beta_{1,2}^* & 0 & \cdots & \cdots & 0 & \beta_{2,n} \\ \vdots & \vdots & \cdot & \cdot & \vdots & \vdots \\ \vdots & \vdots & \cdot & \cdot & \vdots & \vdots \\ \beta_{1,n-1}^* & 0 & \cdots & \cdots & 0 & \beta_{n-1,n} \\ 0 & \beta_{2,n}^* & \cdots & \cdots & \beta_{n-1,n}^* & 0 \end{bmatrix} \quad (526)$$

We may write the quantum brachistochrone as the form:

$$i \frac{d}{dt} (\tilde{H}[t] + \tilde{F}[t]) \in G_F \quad (527)$$

Hence the Hamiltonian is constant:

$$\frac{d}{dt} \tilde{H}[t] = 0 \quad (528)$$

Note that, by induction, we have shown the existence of a geodesic on $\text{SU}(n)$, having shown the existence of geodesics on $\text{SU}(2)$, $\text{SU}(2+1)$, $\text{SU}(n)$ and $\text{SU}(n+1)$.

Another family of infinite dimensional matrices is given by:

$$\tilde{H} = \begin{bmatrix} 0 & \varepsilon_1 & 0 & \cdots & \cdots & 0 \\ \varepsilon_1^* & 0 & \varepsilon_2 & 0 & \cdots & \vdots \\ 0 & \varepsilon_2^* & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \varepsilon_{n-1} & 0 \\ \vdots & \cdots & 0 & \varepsilon_{n-1}^* & 0 & \varepsilon_n \\ 0 & \cdots & \cdots & 0 & \varepsilon_n^* & 0 \end{bmatrix} \quad (529)$$

with associated constraint matrix $\tilde{F} =$

$$\begin{bmatrix} \omega_1 & 0 & \beta_{1,3} & \beta_{1,4} & \cdots & \beta_{1,n-1} & \beta_{1,n} \\ 0 & \omega_2 & 0 & \beta_{2,4} & \cdots & \cdots & \beta_{2,n} \\ \beta_{1,3}^* & 0 & \omega_3 & 0 & \cdots & \cdots & \vdots \\ \beta_{1,4}^* & \beta_{2,4}^* & 0 & \ddots & \ddots & \beta_{n-3,n-1} & \beta_{n-3,n} \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & \beta_{n-2,n} \\ \beta_{1,n-1}^* & \vdots & \vdots & \beta_{n-3,n-1}^* & 0 & \omega_{n-1} & 0 \\ \beta_{1,n}^* & \beta_{2,n}^* & \cdots & \beta_{n-3,n}^* & \beta_{n-2,n}^* & 0 & \omega_n \end{bmatrix} \quad (530)$$

Again, using the quantum brachistochrone, we find that the Hamiltonian matrix, which defines a continuous symmetry group, is constant. One more infinite dimensional system has system Hamiltonian and constraint:

$$\tilde{H}[t] = \begin{bmatrix} \omega_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \omega_2 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & \omega_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 & \omega_n \end{bmatrix} \quad (531)$$

$$\tilde{F}[t] = \begin{bmatrix} 0 & \eta_{1,2} & \cdots & \cdots & \cdots & \eta_{1,n} \\ \eta_{1,2}^* & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \eta_{n-1,n} \\ \eta_{1,n}^* & \cdots & \cdots & \cdots & \eta_{n-1,n}^* & 0 \end{bmatrix} \quad (532)$$

$$\sum_j \omega_j(t) = \omega_1 + \omega_2 + \cdots + \omega_j + \cdots + \omega_{n-1} + \omega_n = 0 \quad (533)$$

Obviously, by construction, the Hamiltonian commutes with the constraint, and is therefore constant. The difficulty with these problems on arbitrary dimensions is that there is no obvious way in which the Lagrange multipliers, representing total upper bounds on the energy, generalises to a matrix of infinite dimensions. One would hope that the sums would go over to integrals in the correct fashion, but this remains to be shown. Of course it is possible to create these type of general Hamiltonian-constraint systems with the geodesic property, but it would be desirable to have expressions for field densities for matrices of infinite dimension, which represent the continuous degrees of freedom.

In conclusion, we have shown within this paper how one might go about calculating these more difficult examples on finite dimensional systems, as well as demonstrating the links between the matrix and vector dynamic calculus. This paper has demonstrated that the use of finite matrix methods and algebraic techniques may yield dividends, even when the situation is graphically complex. We have used the quantum brachistochrone equation to derive a number of new results on $SU(3)$; these methods may be extended to higher-dimensional matrix groups in order to find time-optimal flows of quantum states. The results derived are intriguing, especially the links between the various branches of mathematics. Our principal results may be applied within the laboratory to achieve fast data transfer; as the method of calculation is not reliant on adiabatic transfer or the rotating wave approximation it is theoretically the most general and efficient scheme that may be applied.

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