Time Optimal Control on SU(4)

$$i\frac{d}{dt}(\tilde{H}+\tilde{F}) = \tilde{H}\tilde{F} - \tilde{F}\tilde{H} \tag{1}$$

$$\tilde{H} = \begin{bmatrix} \alpha & 0 & -i\gamma & \beta^* \\ 0 & \alpha & -\beta & i\gamma \\ i\gamma & -\beta^* & -\alpha & 0 \\ \beta & -i\gamma & 0 & -\alpha \end{bmatrix} \tag{2}$$

$$\tilde{F} = \begin{bmatrix} \omega_1 & \epsilon_1^* & \epsilon_2^* & \epsilon_3^* \\ \epsilon_1 & -\omega_1 & \epsilon_3 & \epsilon_4^* \\ \epsilon_2 & \epsilon_3^* & \omega_2 & \epsilon_5^* \\ \epsilon_3 & \epsilon_4 & \epsilon_5 & -\omega_2 \end{bmatrix} \tag{3}$$

$$Tr(\tilde{H}^2/2) = constant = 2(\alpha^2 + |\beta|^2 + \gamma^2) \tag{4}$$

By construction (breaking it up over orthogonal generators of SU(4)):

$$Tr(\tilde{H}\tilde{F}) = -i\gamma(\epsilon_2^* - \epsilon_4^* - \epsilon_2 + \epsilon_4) = 0 \tag{5}$$

The differential equations from (1) read as:

$$i\frac{d}{dt}(\alpha + \omega_1) = -i\gamma(\epsilon_2 + \epsilon_2^*) + \beta^*\epsilon_3^* - \beta\epsilon_3 \tag{6}$$

$$i\frac{d}{dt}(\epsilon_1^*) = \beta^*(\epsilon_2 + \epsilon_4^*) \tag{7}$$

$$i\frac{d}{dt}(\epsilon_2^* - i\gamma) = 2\alpha\epsilon_2 + i\gamma(\omega_1 - \omega_2) + \beta\epsilon_1 + \beta^*\epsilon_5 \tag{8}$$

$$i\frac{d}{dt}(\epsilon_3^* + \beta^*) = 2\alpha\epsilon_3 - i\gamma(\epsilon_5 + \epsilon_1) - \beta^*(\omega_1 + \omega_2) \tag{9}$$

$$i\frac{d}{dt}(\epsilon_1) = -\beta(\epsilon_4 + \epsilon_2^*) \tag{10}$$

$$i\frac{d}{dt}(\alpha - \omega_1) = i\gamma(\epsilon_4 + \epsilon_4^*) + \beta^*\epsilon_3^* - \beta\epsilon_3 \tag{11}$$

$$i\frac{d}{dt}(\epsilon_3 - \beta) = -\beta(\omega_1 + \omega_2) + i\gamma(\epsilon_5^* + \epsilon_1^*) + 2\alpha\epsilon_3^* \tag{12}$$

$$i\frac{d}{dt}(\epsilon_4^* + i\gamma) = 2\alpha\epsilon_4 + i\gamma(\omega_1 - \omega_2) - \beta\epsilon_5 - \beta^*\epsilon_1 \tag{13}$$

$$i\frac{d}{dt}(\epsilon_2 + i\gamma) = i\gamma(\omega_1 - \omega_2) - 2\alpha\epsilon_2^* - \beta\epsilon_5 - \beta^*\epsilon_1 \tag{14}$$

$$i\frac{d}{dt}(\epsilon_3^* - \beta^*) = \beta(\omega_1 + \omega_2) - 2\alpha\epsilon_3 + i\gamma(\epsilon_5 + \epsilon_1) \tag{15}$$

$$i\frac{d}{dt}(\omega_2 - \alpha) = i\gamma(\epsilon_2 + \epsilon_2^*) + \beta\epsilon_3 - \beta^*\epsilon_3^* \tag{16}$$

$$i\frac{d}{dt}(\epsilon_5^*) = -\beta^*(\epsilon_4 + \epsilon_2^*) \tag{17}$$

$$i\frac{d}{dt}(\epsilon_3 + \beta) = \beta(\omega_1 + \omega_2) - i\gamma(\epsilon_5^* + \epsilon_1^*) - 2\alpha\epsilon_3^* \tag{18}$$

$$i\frac{d}{dt}(\epsilon_4 - i\gamma) = i\gamma(\omega_1 - \omega_2) + \beta\epsilon_1 + \beta^*\epsilon_5 - 2\alpha\epsilon_4^* \tag{19}$$

$$i\frac{d}{dt}(\epsilon_5) = \beta(\epsilon_2 + \epsilon_4^*) \tag{20}$$

$$i\frac{d}{dt}(-(\omega_2 + \alpha)) = -i\gamma(\epsilon_4 + \epsilon_4^*) + \beta\epsilon_3 + \beta^*\epsilon_3^* \tag{21}$$

The constants of motion I have managed to find so far:

$$(\omega_1 + \omega_2) = constant \tag{22}$$

$$\epsilon_3^* = const \tag{23}$$

$$\epsilon_3 = const \tag{24}$$

What is the best way of going about solving this sort of problem?