Elliptic Hamiltonian

Choose a Hamiltonian operator and orthogonal constraint in SU(3) defined by the matrices:

$$\tilde{H}(t)=\begin{pmatrix}0 & \alpha(t) & 0\\ \alpha(t) & 0 & -i\beta(t)\\ 0 & i\beta(t) & 0\end{pmatrix}\qquad \tilde{F}(t)=\begin{pmatrix}\omega_1(t) & -i\gamma(t) & \varepsilon(t)\\ i\gamma(t) & \omega_2(t) & \kappa(t)\\ \varepsilon^*(t) & \kappa(t) & \omega_3(t)\end{pmatrix}$$

where $\alpha,\beta,\gamma,\kappa,\omega_1,\omega_2,\omega_3$ are real functions of time and $\varepsilon(t)$ is a complex function of time.

These matrix operators obey the algebraic relations $\mathrm{Tr}(\tilde{H}\tilde{F})=\mathrm{Tr}(\tilde{H})=\mathrm{Tr}(\tilde{F})=0$; we also have an isotropic constraint, given by $\mathrm{Tr}(\tilde{H}^2/2)=\alpha^2(t)+\beta^2(t)=\mathrm{const.}$

We have the quantum brachistochrone equation:

$$i\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{H}(t)+\tilde{F}(t))=\tilde{H}(t)\tilde{F}(t)-\tilde{F}(t)\tilde{H}(t)$$

Making the definition $\varepsilon(t)=\Omega(t)-iv(t)$ we obtain the system of differential equations given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}=2\begin{pmatrix}\gamma\alpha\\ -(\gamma\alpha+\beta\kappa)\\ \beta\kappa\end{pmatrix};\qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\gamma\\ \kappa\end{pmatrix}=\begin{pmatrix}-\beta v-(\omega_1-\omega_2)\alpha\\ -\alpha v+(\omega_2-\omega_3)\beta\end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=-i\Omega\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix};\qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\Omega\\ v\end{pmatrix}=\begin{pmatrix}0\\ \alpha\kappa+\beta\gamma\end{pmatrix}$$

which has a solution for the time-dependent Hamiltonian, given by the matrix:

$$\tilde{H}_{\mathrm{opt}}(t)=R\begin{pmatrix}0 & \cos(\Omega t+\theta) & 0\\ \cos(\Omega t+\theta) & 0 & -i\sin(\Omega t+\theta)\\ 0 & +i\sin(\Omega t+\theta) & 0\end{pmatrix}$$

Take a wave-vector, described by a superposition of time-dependent eigenvectors:

$$|\psi(t)>=\Delta_1(t)|+R(t)>+\Delta_2(t)|0(t)>+\Delta_3(t)|-R(t)>$$

where

$$\tilde{H}(t)|J(t)>=J|J(t)>$$

Then our final state is given by the time-dependent vector:

$|\psi(t)>=$

$$\begin{pmatrix}\cos(\Omega t+\theta)[z_3\cos(\Omega' t)+(z_1/\Omega')\sin(\Omega' t)]-(\sin(\Omega t+\theta)/\Omega')[i(z_2R/\Omega')+\Omega[(z_1/\Omega')\cos(\Omega' t)-z_3\sin(\Omega' t)]\\ 1/(\Omega'^2)[\Omega z_2+iR(z_1\cos(\Omega' t)-z_3\sin(\Omega' t))]\\ -i[\sin(\Omega t+\theta)[z_3\cos(\Omega' t)+(z_1/\Omega')\sin(\Omega' t)]+(\cos(\Omega t+\theta)/\Omega')[i(z_2R/\Omega')+\Omega[(z_1/\Omega')\cos(\Omega' t)-z_3\sin(\Omega' t)]]]\end{pmatrix}$$

where the parameters are given by the formulae:

$$z_1=\Omega\Delta_2(0)-iR\Delta_-(0);\quad z_2=\Omega\Delta_-(0)-iR\Delta_2(0);\quad z_3=\Delta_+(0)$$

$$\Delta_\pm=\frac{1}{\sqrt{2}}(\Delta_1\pm\Delta_3)\qquad \Omega'=+\sqrt{R^2+\Omega^2}$$

The initial state is given by

$$|\psi(0)>=\begin{pmatrix} z_3\cos\theta - (\sin\theta)[\Omega z_1 + i\,\mathrm{R}\mathrm{z}_2]/\Omega'^2 \\ 1/\Omega'^2[\Omega z_2 + i\,R z_1] \\ -\,i[z_3\sin\theta + (\cos\theta/\Omega'^2)[\Omega z_1 + i\,R z_2]] \end{pmatrix}$$

The question is then, given the form of the Hamiltonian operator as specified by the Quantum Brachistochrne, what is the functional form of the constraint $\tilde{F}(t)$?